TESTING HYPOTHESES ABOUT THE MEAN OF A BIVARIATE NORMAL DISTRIBUTION UNDER RESTRICTED ALTERNATIVES

BY

MOHAMMAD YOUSEF AHMAD AL-RAWWASH

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بسم الله الرحمن الرحيم

«وقل ربي ارحمهما كما ربياني صغيرا»

....اهدي خلاصة جهدي وعملي الى اعز واغلى الناس
....امي وابي

محمد

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Mohammad Yousef Ahmad Al-Rawwash

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ABSTRACT

Testing hypotheses about the mean of a bivariate normal distribution under restricted alternatives

By

Mohammad Yousef Ahmad Al-Rawwash

Let the random vector (X,Y) be distributed as a bivariate normal with mean vector (θ_1,θ_2) and identity covariance matrix. Consider the following testing problem

$P(V)$: H_0: $(\theta_1,\theta_2) = (0,0)$ vs H_1: $(\theta_1,\theta_2) \in V - \{(0,0)\}$

where,

$$V = V_{\beta^*} = \left\{(\Delta,\gamma):\ 0 \le \gamma \le \beta^*\right\} \text{ for } 0 \le \beta^* \le \pi/2$$

where (Δ,γ) is the polar transformation of (θ_1,θ_2) and β^* is the angle of the cone.

For this testing problem , we derive the Likelihood Ratio Test (LRT) $\phi_{V_{\beta^*}}$ and its power function. Also we prove some symmetry and monotonicity properties for the power function.

Furthermore, for the case that $\beta^* = \pi/2$, we construct a test which is admissible for testing the problem $P(V_3)$ and dominates the LRT $\phi_{V_{\beta^*}}$

where,

$$V_3 = \left\{(\Delta,\gamma):\ \Delta \ge 0,\ 0 \le \gamma \le \pi/2\right\}.$$

We give a numerical comparison among nine competitive tests and also a numerical comparison for the power function $\phi_{V_{\beta^*}}$ for different values of β^*.

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بسم الله الرحمن الرحيم

اختبار النسبة الاحتمالية حول وسط التوزيع الطبيعي الثنائي في حالة وجود فرضية بديلة مقيدة

محمد يوسف احمد السرواش

الملخص

تحتوي هذه الرسالة على دراسة ادعاء اختبار النسبة الاحتمالية والذي ينص على ان اقتران القوة يزداد في حالة زيادة القيود على الفرضية البديلة .

تتمثل الفرضية الاساسية في ان المشاهدات قد اخذت من توزيع طبيعي ثنائي ذو وسط مساويا لنقطة الاصل مقابل فرضية بديلة هي ان الوسط مقيد بمخروط ذو جانبين رأسه نقطة الاصل . اشتملت الرسالة على برهنة بعض خصائص اقتران القوة المقابل لاختبار النسبة الاحتمالية المذكور سابقا كما احتوت الرسالة مقارنة عددية لقيم اقترانات القوة المقابلة لتسع من الاختبارات المختلفة عند مستوى الدلالة ٠٫٠٥ لعدة حالات عندما تكون زاوية المخروط ط/٣٠ ، ٢ط/٣٠ ، ... ، ط/٢ . ومن هذه القيم يمكن استنتاج انه ليس بالضرورة ان يزداد اقتران القوة كلما ازدادت القيود على الفرضية البديلة وهذا يناقض الادعاء اعلاه .

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<u>**CONTENTS**</u>

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CHAPTER 1

STATISTICAL MODEL AND STATEMENT OF THE PROBLEM

1.1 Introduction

This chapter presents the statistical model and the problem under consideration. In addition, it contains some definitions and preliminaries needed in the sequel of the thesis. Furthermore, it contains a review of the literature related to the problem.

1.2 The Statistical Model

Let $(X_1, Y_1), (X_2, Y_2), \ldots, (X_n, Y_n)$ be a random sample of size n from a bivariate normal distribution with unknown mean vector (θ_1, θ_2) and identity covariance matrix.

We are interested in a testing problem about the mean vector (θ_1, θ_2), of the form,

$$P(V): H_0: (\theta_1, \theta_2) = (0,0) \text{ vs } H_1: (\theta_1, \theta_2) \in V / \{(0,0)\} \quad (1.2.1)$$

where V is a closed and two-sided cone in $\mathbb{R}^2$ with vertex at (0,0) and each side of the cone is convex. A two-sided cone, V, has the property that if $v \in V$, then $-v \in V$.

In the case of bivariate normal distribution with identity covariance matrix, it is known that $(\sqrt{n}\,\bar{X}, \sqrt{n}\,\bar{Y})$ is a minimal sufficient statistics for (θ_1, θ_2). Therefore without loss of generality, we can concentrate on a random vector (X, Y) which has a bivariate normal distribution with mean (θ_1, θ_2) and identity covariance matrix.

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Now, let V_{β_1}, V_{β_2} be two closed two-sided cones in $\mathbb{R}^2$, with $V_{\beta_1} \subseteq V_{\beta_2}$. Also, let $\phi_{V_{\beta_i}}$ be the Likelihood Ratio Test (LRT) corresponding to $P(V_{\beta_i})$, for $i = 1,2$. The aim of this thesis is to study the behavior of these tests and make a comparison between their powers and the power of some other tests. The cone V_{β_i} can be represented as

$$V_{\beta_i} = \left\{ (\Delta,\gamma): \ 0 \le \gamma \le \beta_i \right\}, \quad 0 \le \beta_i \le \frac{\pi}{2} \quad i = 1,2$$

where (Δ,γ) is the polar transformation of (θ_1,θ_2) and β_2 can allowed to take value 2π, in this case, $V_{\beta_2} = \mathbb{R}^2$.

1.3 Definitions and Preliminaries

In this section we state some definitions and preliminaries which will be needed in the thesis.

First of all, let (ψ, β, P) be a probability space and let X be a random variable with probability density function (p.d.f.) $f(x, \theta)$, $\theta \in \Theta$, where Θ is the parameter space.

In the following we define the concepts domination and admissibility

Definition 1.3.1

Consider the testing problem:

$$H_0: \theta \in \Theta_0 \quad \text{vs} \quad H_1: \theta \in \Theta_1,$$

where Θ_0 and Θ_1 are disjoint subsets of the parameter space Θ.

A test ϕ is said to dominate the test ϕ_0 if

$$E_\theta\, \phi \le E_\theta\, \phi_0, \quad \forall\ \theta \in \Theta_0$$

and

$$E_\theta\, \phi \ge E_\theta\, \phi_0, \quad \forall\ \theta \in \Theta_1,$$

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with strict inequality for some $\theta \in \Theta_0 \cup \Theta_1$.

A test ϕ is said to be inadmissible if there exists another test ϕ^* which dominates it, otherwise ϕ is said to be admissible.

An important and well known concept is the convexity of a set which can be defined as follows.

Definition 1.3.2

A set $C \subset \mathbb{R}^2$ is said to be a convex set if

$$\alpha x + (1-\alpha) y \in C, \quad \forall x, y \in C \text{ and } \alpha \in [0,1].$$

Another concept is the duality of a cone which will be used in this thesis and it can be defined as follows.

Definition 1.3.3

Given a cone $V \in \mathbb{R}^2$, the dual cone V^0 of V is defined as

$$V^0 = \left\{ y \in \mathbb{R}^2 : y.v \le 0 \text{ for all } v \in V \right\}.$$

Another important definition which is related to the definition of domination is that of the complete class.

Definition 1.3.4

A class Φ of tests is said to be a complete class if for any test $\phi \notin \Phi$, there exists a test $\phi^* \in \Phi$ which dominates ϕ.

Definition 1.3.5

Let V be some cone with vertex at (0,0). A set $A \subseteq \mathbb{R}^2$ is said to be V-decreasing if for every $x, y \in \mathbb{R}^2$, such that $x \in A$ and $y \le x$ [V], then $y \in A$, where the order relation $y \le x$ [V] means that $(x-y) \in V^0 - \{0\}$.

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Conjecture

Assume that V_1 and V_2 are two subsets of the parameter space Θ, where $V_1 \subset V_2$, then the LRT conjecture can be stated as follows:

Consider the testing problem $P(V_i)$ for $i = 1,2$. Let ϕ_i be the corresponding LRT's. It is conjectured that the LRT ϕ_1 dominates ϕ_2 (in terms of the power function).

The following theorem will give us an idea about the conditions under which a test belongs to a minimal complete class in case of an exponential family, and certain types of hypotheses.

Theorem 1.3.1 *(Eaton's Theorem 1970)*

Suppose that X is a random variable with p.d.f

$$f(x,\theta) = A(x)\, B(\theta)\, e^{x\theta} \qquad \text{where} \qquad (x,\theta) \in \mathbb{R}^2$$

We want to test

$$H_0:\ (\theta_1,\theta_2) = (0,0) \qquad \text{vs} \qquad H_1:\ (\theta_1,\theta_2) \in V \setminus \{(0,0)\}$$

where V is a closed convex cone in $\mathbb{R}^2$ with vertex (0,0). Then the class of all tests which have a convex and V-decreasing acceptance region forms a minimal complete class.

Any test which satisfies Eaton's conditions will belong to a minimal complete class and therefore will be an admissible test.

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1.4 Review of the literature

This section presents a summary of the literature related to the testing problem under consideration.

Bartholomew (1959a) considered testing hypothesis about the mean vector of a multivariate normal distribution against an ordered alternative. He derived the LRT for this testing problem.

Later, Bartholomew (1959b, 1961) considered the same testing problem and gave an additional properties of the null distribution of the LRT. Particularly, he showed that the distribution depends on the usual χ^2-distribution and certain probabilities. He also obtained the power function of the LRT for some special cases. He concluded that the use of the restricted alternative leads to an increase in the power of the LRT.

Kudô (1963) and Nuesch (1966) considered the testing problem

$$H_0: \theta_i = 0 \quad \text{vs} \quad H_1: \theta_i > 0, \text{ for } i = 1, 2, \ldots, p,$$

where $\theta_1, \theta_2, \ldots, \theta_p$ are the means of p-variate normal distribution with known covariance matrix. It can be seen that the alternative space is the non-negative orthant. Both Kudô and Nuesch independently obtained the LRT and its distribution under the null hypotheses.

Kudô and Choi (1975) continued the work of Kudô and Nuesch. Instead of a closed convex cone they considered as

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alternative space a convex polyhedral cone determined by linear inequalities. This can be considered as a multivariate generalization of the one-sided test. For this testing problem they derived the LRT and its null distribution and compared the power of the LRT with power of the usual χ^2-test.

Shirahata (1977), considered the testing problem

$$H_0: \theta_1 = \theta_2 = 0 \quad \text{vs} \quad H_1: \theta_1 > 0,\ \theta_2 > 0,$$

for a bivariate normal distribution with mean vector (θ_1, θ_2) and a known covariance matrix. He suggested a new testing procedure which is called the Likelihood Ratio Conditional Probability Principle Test (LRCPPT). He conjectured that the LRT is not optimal in general. He showed that when the population correlation is positive, the LRCPPT is not good, i.e. the power of the LRCPPT is much less than the power of the LRT. In the case where the population correlation coefficient is negative the LRCPPT is more powerful than the LRT.

Sasabuchi (1980), considered a p-variate normal distribution with mean vector $(\theta_1, \theta_2, \ldots, \theta_p)$ and known covariance matrix. He considered the testing problem, where the null hypotheses is that θ lies on the boundary of a convex polyhedral cone determined by linear inequalities, and the alternative is that θ lies in its interior. He derived the LRT and illustrated some of its applications. Also, he discussed the geometry of convex polyhedral cones.

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Al-Rawwash (1986), considered the testing problem

H_0: $(\theta_1, \theta_2) = (0,0)$ vs H_1: $(\theta_1, \theta_2) \in V_i - \{(0,0)\}$, $i = 1,2$

He derived the LRT for three special cases namely

$V_1 = \bar{\mathbb{R}}$; $\bar{\mathbb{R}} = [0,\infty)$ $\quad$ $V_2 = \mathbb{R}^2$,

$V_1 = \bar{\mathbb{R}} \times \mathbb{R}$ $\quad$ $V_2 = \mathbb{R}^2$ and

$V_1 = \bar{\mathbb{R}}^2$ $\quad$ $V_2 = \bar{\mathbb{R}} \times \mathbb{R}$.

Also, he showed that the power function of the LRT satisfies certain monotonicity properties.

Al-Rawwash, and Marden (1988), developed a new method for the construction of a test which is admissible and strictly dominates a given inadmissible one.

Al-Shrouf (1989), considered the testing problem

H_0: $(\theta_1, \theta_2) = (0,0)$ vs H_1: $(\theta_1, \theta_2) \in V_i - \{(0,0)\}$, $i = 1,2$

for certain two cones V_1 and V_2 where, $V_1 \subseteq V_2$.

He showed that the LRT ϕ_{V_1} dominates the LRT ϕ_{V_2} in terms of the power function.

Odibat (1989), considered the testing problem

H_0: $(\theta_1, \theta_2) = (0,0)$ vs H_1: $(\theta_1, \theta_2) \in V_i - \{(0,0)\}$, $i = 1,2$

for certain two cones V_1 and V_2 where, $V_1 \subseteq V_2$.

He proved the LRT conjecture for the case V_1 a closed convex cone with vertex at (0,0) and angle less than $\frac{\pi}{2}$ and $V_2 = \mathbb{R}^2$.

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1.5 Summary of the thesis

This thesis is concerned with the LRT conjecture. It's shown, through numerical computation, that if we have a restriction on the alternative space then it is not necessarily true that the power function will increase.

In chapter one, we describe the statistical model and the testing problems that we wish to study. Also, a brief historical review of the related literature is given. Some important concepts and theorems that will help us in studying these testing problems are also stated in chapter 1.

Chapter two, gives a derivation of the LRT and its distribution under the null hypothesis. Also, the power function of the LRT is obtained. Furthermore, two properties of the power function are given namely, symmetry and monotonicity.

Chapter three contains a construction of a test that dominates the inadmissible LRT ϕ_{v_1}. We use the method of Al-Rawwash and Marden (1988) to construct such a test. Furthermore, this chapter contains some numerical comparisons of the power values among nine competitive tests.

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Chapter 2

Derivation of the LRT and its Power Function

2.1 Introduction

Consider the statistical model given in (1.2.1) in which the random vector (X, Y) is distributed as a bivariate normal with mean vector (θ_1, θ_2) and identity covariance matrix.

In this chapter, the LRT for the testing problem given in (1.2.1) is described. In addition, a derivation for the power function associated with the LRT is obtained. Also, some properties of the power function viz., symmetry and monotonocity is given.

The following section is devoted to the derivation of the LRT for the testing problem given in (1.2.1).

2.2 Derivation of the LRT

To derive the LRT we have to find the MLE restricted to V. To accomplish this we dived the sample space $\mathbb{R}^2$ into five regions. These regions are

$$V = \left\{ (\Delta,\gamma): \Delta \geq 0;\ 0 \leq \gamma \leq \beta^* \text{ or } \pi \leq \gamma \leq \pi + \beta^* \right\},$$

$$V_1^+ = \left\{ (\Delta,\gamma): \Delta \geq 0;\ \beta^* \leq \gamma \leq \frac{\beta^* + \pi}{2} \right\},$$

$$V_2^+ = \left\{ (\Delta,\gamma): \Delta \geq 0;\ \frac{\beta^* + \pi}{2} \leq \gamma \leq \pi \right\},$$

$$V_1^- = \left\{ (\Delta,\gamma): \Delta \geq 0;\ \beta^* + \pi \leq \gamma \leq \frac{3\pi + \beta^*}{2} \right\},$$

$$V_2^- = \left\{ (\Delta,\gamma): \Delta \geq 0;\ \frac{3\pi + \beta^*}{2} \leq \gamma \leq 2\pi \right\},$$

where, β^* is the angle of cone V.

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These regions are illustrated in Figure 2.2.1.

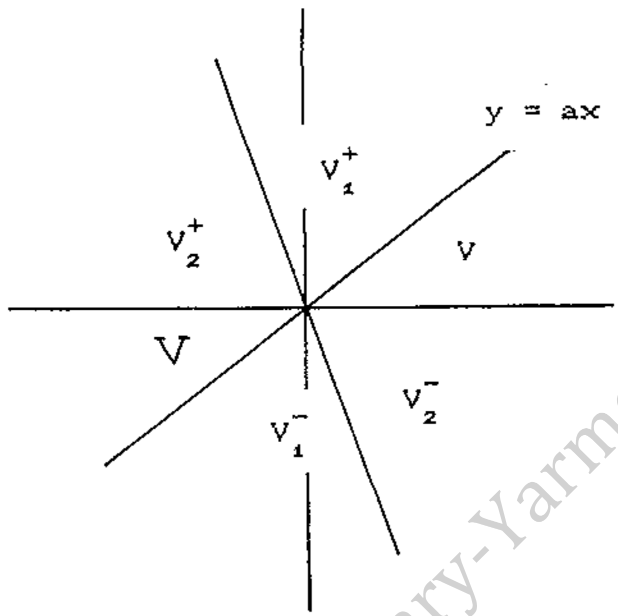

Figure 2.2.1 : The partitions of the space $\mathbb{R}^2$

In the following theorem we present the MLE of (θ_1, θ_2) restricted to V.

Theorem 2.2.1

The MLE of (θ_1, θ_2) restricted to V is given by

$$(\hat{\theta}_1, \hat{\theta}_2) = \begin{cases} (x, y) & \text{if } (x,y) \in V \\ \left(\dfrac{x + a\,y}{1 + a^2},\ a\,\dfrac{x + a\,y}{1 + a^2}\right) & \text{if } (x,y) \in V_1^+ \text{ or } V_1^- \\ (x, 0) & \text{if } (x,y) \in V_2^+ \text{ or } V_2^- \end{cases} \tag{2.2.1}$$

where $a = \tan(\beta^*)$.

Proof:

Consider the random vector (X,Y) which is distributed as a bivariate normal with mean vector (θ_1, θ_2) and identity

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covariance matrix. Therefore, the *pdf* can be written as

$$f(x,y,\theta_1,\theta_2) = \frac{1}{2\pi} \exp\left\{-\frac{1}{2}(x-\theta_1)^2 - \frac{1}{2}(y-\theta_2)^2\right\}.$$

Let $(\hat\theta_1,\hat\theta_2)$ be the point which minimize

$$\Delta^2 = (x-\theta_1)^2 + (y-\theta_2)^2 \quad \text{for all } (\theta_1,\theta_2) \in V$$

Therefore, $(\hat\theta_1,\hat\theta_2)$ will be the closest point to (x,y). Now, if $(x,y) \in V$ then $(\hat\theta_1,\hat\theta_2) = (x,y)$. But, if $(x,y) \notin V$, then (x,y) will belong to one of regions V_1^+, V_2^+, V_1^- and V_2^-.

If $(x,y) \in V_2^-$, the MLE will be in the set $\{(t,0) : t \geq 0\}$, and it can be seen that the closest point to V is $(x,0)$ which is the MLE of (θ_1,θ_2). Also, if $(\theta_1,\theta_2) \in V_1^+$, then the MLE will be in the set $\{(t,u) : u = at\}$, and $a = \tan\beta^*$ which implies that $\theta_2 = a\theta_1$, therefore

$$\Delta^2 = (x-\theta_1)^2 + (y-a\theta_1)^2.$$

Now,

$$\frac{\partial \Delta^2}{\partial \theta_1} = -2(x-\theta_1) - 2a(y-a\theta_1)$$

and

$$\left.\frac{\partial \Delta^2}{\partial \theta_1}\right|_{\theta_1=\hat\theta_1} = 0 \quad \text{implies that}$$

$$\hat\theta_1 = \frac{x+ay}{1+a^2} \quad \text{and} \quad \hat\theta_2 = a\,\frac{x+ay}{1+a^2}.$$

In the same manner, we can see that if $(\theta_1,\theta_2) \in V_2^+$ then the MLE $(\hat\theta_1,\hat\theta_2) = (x,0)$, and if $(\theta_1,\theta_2) \in V_1^-$ then the MLE

$(\hat\theta_1,\hat\theta_2) = \left(\frac{x+ay}{1+a^2}, a\,\frac{x+ay}{1+a^2}\right)$. This completes the proof.

We use the above result to derive the LRT as follows

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Theorem 2.2.2

The LRT for the testing problem (1.2.1) is given by

$$\phi_{V_{\beta^*}} = \begin{cases} 1 & \text{if } \bar{\chi}^2 > k \\ 0 & \text{if } \bar{\chi}^2 \le k \end{cases},$$

where,

$$\bar{\chi}^2 = \begin{cases} x^2 + y^2 & \text{if } (x,y) \in V \\ x^2 & \text{if } (x,y) \in V_2^- \text{ or } V_2^+ \\ \dfrac{(x + a\,y)^2}{1 + a^2} & \text{if } (x,y) \in V_1^- \text{ or } V_1^+ \end{cases}. \qquad (2.2.2)$$

Proof:

Assume that (X,Y) is a random vector distributed as bivariate normal with mean vector (θ_1,θ_2) and identity covariance matrix. The *pdf* can be written as

$$f(x,y,\theta_1,\theta_2) = \frac{1}{2\,\pi}\, e^{-\frac{1}{2}\left\{(x-\theta_1)^2+(y-\theta_2)^2\right\}}$$

The LRT statistic Λ is given by

$$\Lambda = \sup_{(\theta_1,\theta_2)\in\Theta_0} f(x,y;\theta_1,\theta_2) \Big/ \sup_{(\theta_1,\theta_2)\in\Theta_1} f(x,y;\theta_1,\theta_2)$$

$$= \frac{e^{-\frac{1}{2}(x^2+y^2)}}{\sup\limits_{(\theta_1,\theta_2)\in\Theta_1} e^{-\frac{1}{2}\{(x-\theta_1)^2+(y-\theta_2)^2\}}}$$

$$= \operatorname{Exp}\left\{-\frac{1}{2}\inf_{(\theta_1,\theta_2)\in\Theta_1}\{x^2+y^2-(x-\theta_1)^2-(y-\theta_2)^2\}\right\}$$

$$= \operatorname{Exp}\{-\frac{\bar{\chi}^2}{2}\},$$

where,

$$\bar{\chi}^2 = \{x^2+y^2-(x-\hat{\theta}_1)^2-(y-\hat{\theta}_2)^2\}$$

where $(\hat{\theta}_1,\hat{\theta}_2)$ is the MLE of (θ_1,θ_2) restricted to V which is given in (2.2.1). This completes the proof.

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Figure 2.2.2 illustrates the acceptance region for the LRT of the testing problem (1.2.1).

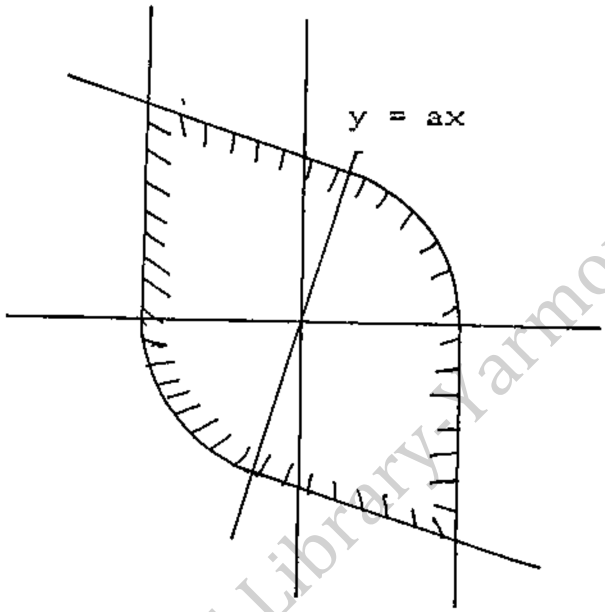

Figure 2.2.2 : Acceptance region of the test $\phi_{V_{\beta^*}}$

2.3 Derivation of the Power Function

In the previous section we derived the LRT and showed that it is given by (2.2.2). Here we present the derivation of the power function of the LRT. In addition, we will state few theorems that will reveal some properties of the power function.

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Theorem 2.3.1

Consider the LRT given in (2.2.2). For any $(\theta_1, \theta_2) \in V$ let (Δ, γ) be the polar transformation of (θ_1, θ_2). Then the power function of the LRT can be expressed as

$$\beta(\Delta,\gamma) = \beta_1(\Delta,\gamma) + \beta_2(\Delta,\gamma) + \beta_3(\Delta,\gamma) - \beta_4(\Delta,\gamma)$$

where,

$$\beta_1(\Delta,\gamma) = \frac{e^{-\frac{1}{2}\Delta^2}}{2\pi} \left[\int_{-\gamma}^{\beta^*-\gamma} (H_1(k,\Delta\cos\xi) + H_2(k,\Delta\cos\xi))\, d\xi \right],$$

$$\beta_2(\Delta,\gamma) = F(\gamma) + F(\beta^*-\gamma),$$

$$\beta_3(\Delta,\gamma) = G(\gamma) + G(\beta^*-\gamma),$$

$$\beta_4(\Delta,\gamma) = \int_{\sqrt{k}}^{\infty} \int_{-\infty}^{-\frac{1}{a}(x+\sqrt{k(1+a^2)})} (Z(x-\Delta\cos\gamma)Z(y-\Delta\sin\gamma) + Z(x+\Delta\cos\gamma)Z(y+\Delta\sin\gamma))\, dy$$

$$H_1(k,t) = \int_{\sqrt{k}}^{\infty} re^{-\frac{1}{2}r^2+rt}dt\,, \quad H_2(k,t) = H_1(k,-t)$$

$$F(\gamma) = Q(\Delta\sin(-\gamma))\, Q(\sqrt{k} + \Delta\cos\gamma),$$

$$G(\gamma) = Q(\Delta\sin\gamma)\, Q(\sqrt{k} - \Delta\cos\gamma),$$

$$Z(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \quad \text{and} \quad Q(x) = \int_x^{\infty} Z(t)dt.$$

Proof:

The proof of the theorem is based on dividing the rejection region into subregions as follows:

$$A_1 = \left\{ (\Delta,\gamma):\ \Delta \geq \sqrt{k}\ ;\ 0 \leq \gamma \leq \beta^* \right\},$$

$$A_2 = \left\{ (x,y):\ x \geq \sqrt{k};\ y \leq 0 \right\},$$

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$A_3 = \{ (x,y): x \le -q \ \& \ y \le ax \} \cup \{ (x,y): x \ge -q \ \& \ y \le L_1 \}$

$A_4 = \{ (\Delta,\gamma): \Delta \ge \sqrt{k}; \quad \pi \le \gamma \le \pi + \beta^{*'} \}$,

$A_5 = \{ (x,y): x \le -\sqrt{k}; \ y \ge 0 \}$,

$A_6 = \{ (x,y): x \ge q \ \& \ y \ge ax \} \cup \{ (x,y): x \le q \ \& \ y \ge L_2 \}$

where, $q = \dfrac{\sqrt{k}}{\sqrt{1 + a^2}}$,

$L_1 = -\frac{1}{a} \left[x + \sqrt{k(1 + a^2)} \right]$ and

$L_2 = -\frac{1}{a} \left[x - \sqrt{k(1 + a^2)} \right]$.

These regions are shown in Figure 2.3.1.

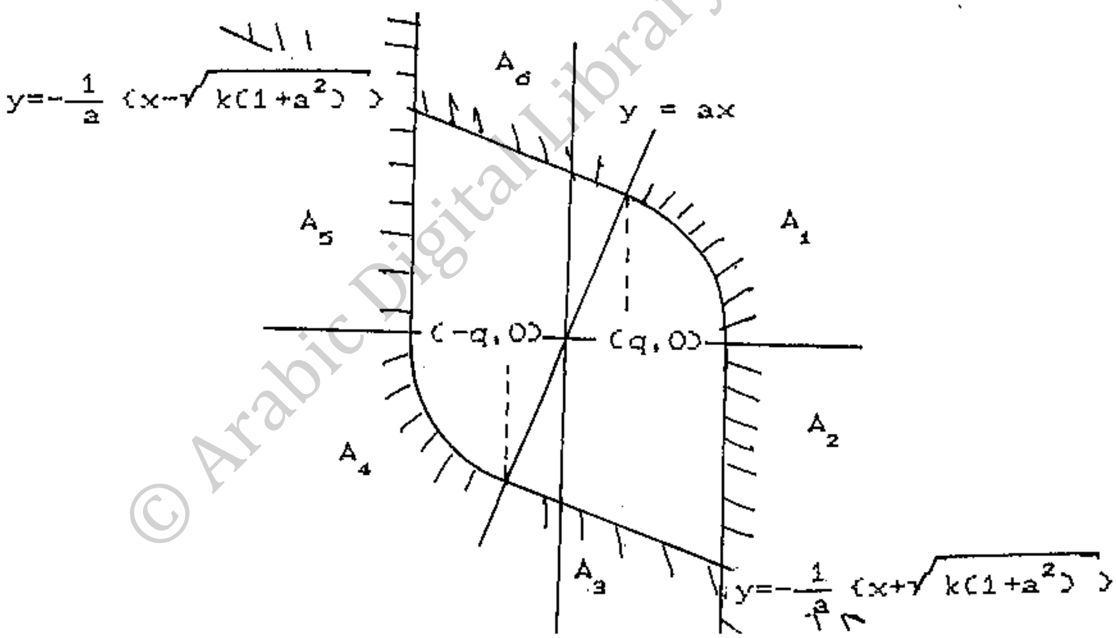

Figure 2.3.1 : Subregions of the rejection region

Hence,

$$\beta(\Delta,\gamma) = \Pr(\text{ rej. } H_0 \mid \text{The true value is } (\Delta,\gamma)) = \sum_{i=1}^{6} \Pr(A_i).$$

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But,

$$Pr(A_1) = \int_{A_1}\int Z(x-\theta_1)\, Z(y-\theta_2)\, dy\, dx$$

$$= \int_{A_1}\int \frac{1}{2\pi} \exp\left\{ -\frac{1}{2}(x-\theta_1)^2 - \frac{1}{2}(y-\theta_2)^2 \right\} dy\, dx$$

$$= \int_0^{\beta^*}\int_{\sqrt{K}}^{\infty} \frac{e^{-\frac{1}{2}\Delta^2}}{2\pi}\, r \exp\left\{ -\frac{1}{2}(r^2 - 2r\Delta\cos(\xi-\gamma)) \right\} dr\, d\xi.$$

Let $\xi_1 = \xi - \gamma$, then

$$Pr(A_1) = \int_{-\gamma}^{\beta^*-\gamma}\int_{\sqrt{K}}^{\infty} \frac{e^{-\frac{1}{2}\Delta^2}}{2\pi}\, r \exp\left\{ -\frac{1}{2}(r^2 - 2r\Delta\cos\xi_1\ ) \right\} dr\, d\xi_1$$

$$= \frac{e^{-\frac{1}{2}\Delta^2}}{2\pi}\int_{-\gamma}^{\beta^*-\gamma} H_1(k,\ \Delta\cos\xi_1)\, d\xi_1.$$

Similarly,

$$Pr(A_4) = \frac{e^{-\frac{1}{2}\Delta^2}}{2\pi}\int_{-\gamma}^{\beta^*-\gamma} H_2(k,\ \Delta\cos\xi_1)\, d\xi_1.$$

Also,

$$Pr(A_2) = \int_{\sqrt{K}}^{\infty}\int_{-\infty}^{0} Z(x-\theta_1)\, Z(y-\theta_2)\, dy\, dx$$

$$= \int_{\sqrt{K}}^{\infty} Z(x-\theta_1)\, N(y-\theta_2)\Big|_{-\infty}^{0}\, dx$$

$$= Q(\theta_2)\, N(x-\theta_1)\Big|_{\sqrt{K}}^{\infty}$$

$$= Q(\theta_2)\, Q(\sqrt{K} - \theta_1)$$

In view of the polar transformation (Δ,γ) of (θ_1,θ_2) we get

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$$P(A_2) = Q(\Delta \sin\gamma)\, Q(\sqrt{K} - \Delta \cos\gamma)$$
$$= G(\gamma).$$

In the same manner,

$$Pr(A_5) = \int_{-\infty}^{-\sqrt{K}} \int_{0}^{\infty} Z(x-\theta_1)\, Z(y-\theta_2)\, dy\, dx$$
$$= Q(\Delta \sin(-\gamma))\, Q(\sqrt{K} + \Delta \cos\gamma)$$
$$= F(\gamma).$$

In order to compute $P(A_3)$ and $P(A_6)$ we make a transformation of the xy -plane into $x'y'$-plane which is obtained by rotating through an angle β^* clockwise, i.e

$$X' = X \cos\beta^* + Y \sin\beta^*$$
$$Y' = Y \cos\beta^* - X \sin\beta^*$$

In this case, the line $y = ax$ reduces to the line $y' = 0$ which is the x'-axis. This implies that (X', Y') is distributed as a bivariate normal with mean vector $(\mu_{X'}, \mu_{Y'})$ and identity covariance matrix, with

$$\mu_{X'} = \theta_1 \cos\beta^* + \theta_2 \sin\beta^*$$
$$= \Delta\cos\gamma \cos\beta^* + \Delta\sin\gamma \sin\beta^*$$
$$= \Delta\cos(\gamma-\beta^*)$$

similarly, $\mu_{Y'} = \Delta\sin(\gamma-\beta^*)$.

It can be easily seen that

$$P(A_3) = N(-\sqrt{K}-\mu_{X'})N(-\mu_{Y'})$$
$$= Q(\sqrt{K}+\Delta os(\beta^*-\gamma))Q(-\Delta\sin(\beta^*-\gamma))$$
$$= F(\beta^*-\gamma)$$

similarly, $P(A_6) = G(\beta^*-\gamma)$

Now,

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$$Pr(A_2 \cap A_3) = \int_{\sqrt{k}}^{\infty} \int_{-\infty}^{-\frac{1}{a}(x+\sqrt{k(1+a^2)})} Z(x-\theta_1)\, Z(y-\theta_2)\, dy\, dx$$

and

$$Pr(A_5 \cap A_6) = \int_{-\infty}^{-\sqrt{k}} \int_{\frac{1}{a}(\sqrt{k(1+a^2)}-x)}^{\infty} Z(x-\theta_1)\, Z(y-\theta_2) dy\, dx.$$

Applying the transformation x = −x, y= −y we get,

$$Pr(A_5 \cap A_6) = \int_{\sqrt{k}}^{\infty} \int_{-\infty}^{-\frac{1}{a}(x+\sqrt{k(1+a^2)})} Z(x+\theta_1)\, Z(y+\theta_2)\, dy\, dx$$

this completes the proof.

The following theorem establishes she symmetry of the power function.

Theorem 2.3.2

For fixed $\Delta \geq 0$, the power function $\beta(\Delta,\gamma)$ is symmetric about $\gamma = \frac{\beta^*}{2}$, i.e. $\beta(\Delta,\ \gamma) = \beta(\Delta,\ \beta^*-\gamma) \quad \forall\ \gamma < \frac{\beta^*}{2}$.

Proof:

We want to show that

$$\beta(\Delta,\ \gamma) = \beta(\Delta,\ \beta^*-\gamma) \quad \forall\ \gamma < \frac{\beta^*}{2},$$

In order to accomplish this, it is enough to show that each $\beta_i(\Delta,\gamma)$, i=1,2,3,4 is symmetric about $\gamma = \beta^*/2$, where β_i's are defined in the statement of theorem 2.3.1

Now,

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$$\beta_1(\Delta,\beta^*-\gamma) = \int_{\gamma-\beta^*}^{\gamma} (H_1(k,\Delta\cos\xi) + H_2(k,\Delta\cos\xi))\, d\xi$$

By transforming ξ to $-\xi$ we get

$$\beta_1(\Delta,\beta^*-\gamma) = \int_{-\gamma}^{\beta^*-\gamma} (H_1(k,\Delta\cos\xi) + H_2(k,\Delta\cos\xi))\, d\xi$$

$$= \beta_1(\Delta,\gamma)$$

Also, it can be easily seen that $\beta_2(\Delta,\gamma)$ and $\beta_3(\Delta,\gamma)$ are symmetric.

In order to complete the proof we need to show that $\beta_4(\Delta,\gamma)$ is symmetric about $\gamma = \frac{\beta^*}{2}$. To accomplish this, transform (x,y) into (x',y') which is obtained by a rotation through an angle $\frac{\beta^*}{2}$ clockwise, i.e.,

$$x' = x\cos\frac{\beta^*}{2} + y\sin\frac{\beta^*}{2} \text{ and } y' = y\cos\frac{\beta^*}{2} - x\sin\frac{\beta^*}{2}.$$

Its easy to see that (X',Y') is distributed as $N((\theta_1',\theta_2'),I)$ where,

$$\theta_1' = \Delta\cos(\gamma - \frac{\beta^*}{2}) \quad \text{and} \quad \theta_2' = \Delta\sin(\gamma - \frac{\beta^*}{2})$$

Now, applying the transformation $\gamma' = \gamma - \frac{\beta^*}{2}$ we get

$$\theta_1' = \Delta\cos(\gamma') \quad \text{and} \quad \theta_2' = \Delta\sin(\gamma')$$

Notic that the regions $(A_2 \cap A_3)$ and $(A_5 \cap A_6)$ in the xy -plane are transformed into M_1 and M_2 in the $x'y'$-plane as shown in figure (2.3.3). Also, if we transform y' to $-y'$ then the region M_1 in the $x'y'$-plane is the same as M_2 in the new $x'y'$-plane.

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Notice that we can write

$$\beta_4(\Delta,\gamma') = \beta_{41}(\Delta,\gamma') + \beta_{42}(\Delta,\gamma')$$

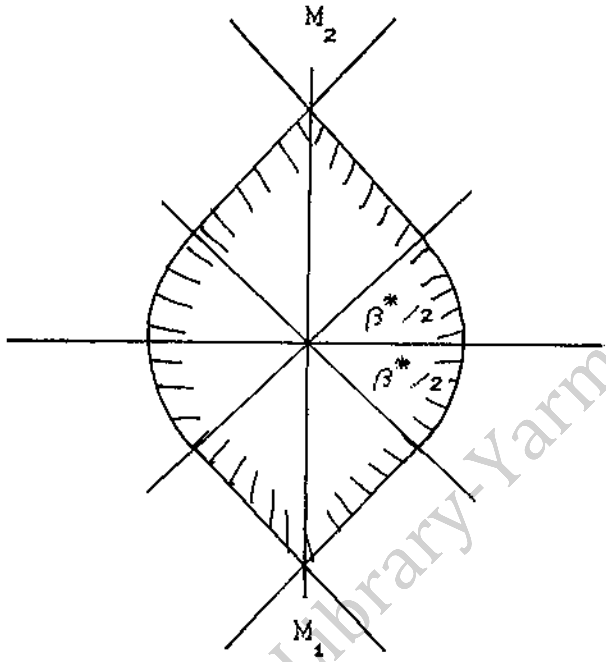

Figure 2.3.2 : Acceptance region after the rotation of angle $\beta^*/2$

where,

$$\beta_{41}(\Delta,\gamma') = \int\int_{M_1} Z(x'-\Delta\cos\gamma')Z(y'-\Delta\sin\gamma')\, dy'dx'.$$

and $$\beta_{42}(\Delta,\gamma') = \int\int_{M_2} Z(x'-\Delta\cos\gamma')Z(y'-\Delta\sin\gamma')\, dy'dx'.$$

But, if we transform y' to -y' we get,

$$\beta_{41}(\Delta,\gamma') = \int\int_{M_2} Z(x'-\Delta\cos\gamma')Z(y'+\Delta\sin\gamma')\, dy'dx'.$$

According to this result we get,

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$$\beta_{41}(\Delta,-\gamma') = \int_{M_2}\int Z(x'-\Delta\cos\gamma')Z(y'-\Delta\sin\gamma')\, dy'dx'.$$

$$= \beta_{42}(\Delta,\gamma') \qquad (2.3.2)$$

But we know that $\beta_4(\Delta,\gamma') = \beta_{41}(\Delta,\gamma') + \beta_{42}(\Delta,\gamma')$, therefore, in view of (2.3.2) we get $\beta_4(\Delta,\gamma')$ is symmetric about $\gamma' = 0$.

Now, we want to show that $\beta_4(\Delta,\gamma')$ is symmetric about $\frac{\beta^*}{2}$.

$$\beta_4(\Delta,\beta^*-\gamma') = \beta_4(\Delta,\beta^*-(\gamma-\frac{\beta^*}{2}))$$
$$= \beta_4(\Delta,\frac{\beta^*}{2}-\gamma)$$
$$= \beta_4(\Delta,-\gamma')$$
$$= \beta_4(\Delta,\gamma')$$

Since $\beta_i(\Delta,\gamma)$, i=1,2,3,4 is symmetric about $\gamma = \frac{\beta^*}{2}$, therefore, $\beta(\Delta,\gamma) = \beta_1(\Delta,\gamma) + \beta_2(\Delta,\gamma) + \beta_3(\Delta,\gamma) - \beta_4(\Delta,\gamma)$ is symmetric about $\gamma = \frac{\beta^*}{2}$. This completes the proof.

The following theorem proves the monotonicity of the power function.

Theorem 2.3.3

The power function of the LRT increases in γ for all $\gamma \le \frac{\beta^*}{2}$ and decreases for all $\gamma \in [\frac{\beta^*}{2},\beta^*]$.

Proof:

Let A^* be the acceptance region of the LRT of the testing problem (1.2.1), then

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$$\beta(\Delta,\gamma) = 1 - \Pr(A^{*})$$

$$= 1 - \iint_{A^{*}} \frac{1}{2\pi} \exp\left\{-\tfrac{1}{2}(x^2+y^2) - \frac{\Delta^2}{2} + x\Delta\cos\gamma + y\Delta\sin\gamma\right\} dx\, dy$$

$$\frac{\partial\beta(\Delta,\gamma)}{\partial\gamma} = -\iint_{A^{*}} \left(\frac{-x\Delta\sin\gamma + y\Delta\cos\gamma}{2\pi}\right) \exp\left\{-\tfrac{1}{2}(x^2+y^2) - \frac{\Delta^2}{2} + x\Delta\cos\gamma + y\Delta\sin\gamma\right\} dx\, dy.$$

Let $X' = X\cos\gamma + Y\sin\gamma$ and $Y' = Y\cos\gamma - X\sin\gamma$, which is a rotation of angle $\gamma \le \frac{\beta^{*}}{2}$ then we have, $\mu_{X'} = \Delta$ and $\mu_{Y'} = 0$, i.e. (X',Y') is distributed as $N((\mu_{X'},\mu_{Y'}),I)$.

Hence,

$$\frac{\partial\beta(\Delta,\gamma)}{\partial\gamma} = -\iint_{A^{**}} \frac{\Delta\, y'}{2\pi} \exp\left\{-\tfrac{1}{2}(x'^2 + y'^2) - \Delta^2/2 + \Delta x'\right\} dx'dy'$$

$$= -\iint_{A^{**}} \frac{\Delta\, y'}{2\pi} \exp\left\{-\tfrac{1}{2}((x'-\Delta)^2 + y'^2)\right\} dx'\, dy'$$

$$= -\iint_{A^{**}} M(x', y')\, dx'\, dy'$$

where,

A^{**} is the acceptance region of the LRT in the $x'y'$-plane.

Now, the acceptance region can be partitioned into four disjoint regions B_1–B_4, which are shown in figure (2.3.3) Notice that if we transform y' into $-y'$ theregion B_2 in the new plane will be the same as B_1 in the old $x'y'$-plane. This can be easily seen from figure (2.3.3).

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Therefore,

$$\int_{B_1}\int M(x',y')\,dx'dy' = -\int_{B_2}\int M(x',y')\,dx'dy'$$

Now, we want to evaluate this integral on the regions B_3 and B_4. Notice that

$$\int_{B_3}\int M(x',y')\,dx'dy' \le 0 \text{ and } \int_{B_4}\int M(x',y')\,dx'dy' \ge 0.$$

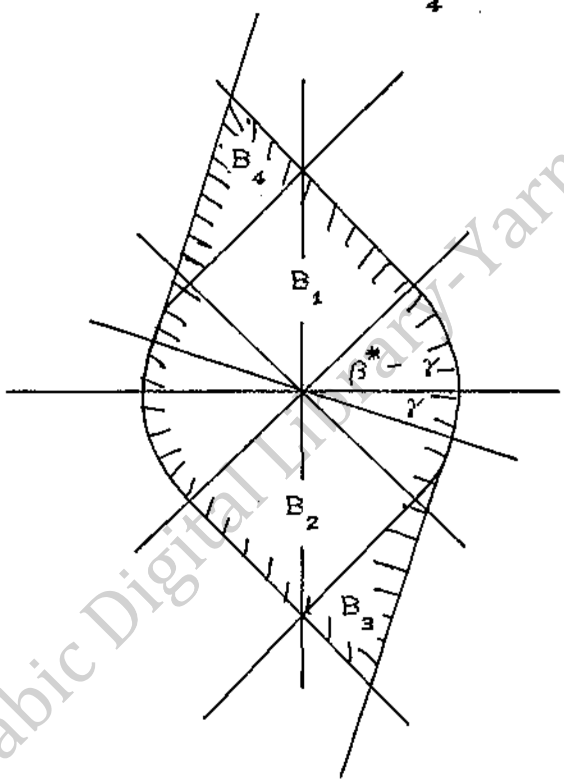

Figure 2.3.3 : Acceptance region after rotation through an angle γ

Transforming (x',y') into $(-x',-y')$, we get that B_4 in the new plane will be the same as B_3 in the old $x'y'$-plane.

Therefore,

$$\int_{B_3}\int M(x',y')\,dx'dy' = \int_{B_4}\int M(-x',-y')\,dx'dy'$$

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$$= - \int_{B_4}\int \frac{\Delta\ y'}{2\ \pi} \exp \left\{ -\frac{1}{2} ((x'+\Delta)^2 + y'^2) \right\} dx'\ dy'$$

$$= - \int_{B_4}\int M^*(x',y')\ dx'dy'$$

where,

$$M^*(x',y') = \frac{\Delta\ y'}{2\ \pi} \exp \left\{ -\frac{1}{2} ((x'+\Delta)^2 + y'^2) \right\}$$

Hence,

$$\int_{B_3}\int M(x',y')\ dx'dy' + \int_{B_4}\int M(x',y')\ dx'dy'$$

$$= \int_{B_4}\int [\ M(x',y') - M^*(x',y')]\ dx'dy'$$

But,

$$M(x',y') - M^*(x',y') = \frac{\Delta\ y'}{2\pi} \exp\ (-\frac{1}{2}y'^2) *$$

$$\left\{ \exp\ [-\frac{1}{2}\ (x'-\Delta)^2] - \exp\ [-\frac{1}{2}\ (x'+\Delta)^2] \right\}$$

Also, on B_4 we can see that $X' \le 0$, $Y' > 0$ and $\Delta \ge 0$,therefore

$$(x'-\Delta)^2 \ \ge (x'+\Delta)^2$$

implies that

$$\exp\ [-\frac{1}{2}\ (x'-\Delta)^2] \le \exp\ [-\frac{1}{2}\ (x'+\Delta)^2]$$

Hence,

$M(x',y') - M^*(x',y') \le 0$ for all $(x',y') \in B_4$.

This means that,

$$\int_{B_3}\int M(x',y')\ dx'dy' + \int_{B_4}\int M(x',y')\ dx'dy' \le 0$$

Therefore,

$$\frac{\partial\beta(\Delta,\gamma)}{\partial\ \gamma} = -\int_{A^{**}}\int M(x',\ y')\ dx'\ dy'$$

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$$= -\int_{B_4}\int [M(x',y') - M^*(x',y')] \, dx'dy' \geq 0$$

which means that $\beta(\Delta,\gamma)$ is increasing for $\gamma < \frac{\beta^*}{2}$.

Corollary 2.3.1

The power function has two equal minima at $\gamma = 0$ and $\gamma = \beta^*$, and it has a maximum at $\gamma = \frac{\beta^*}{2}$.

Theorem 2.3.4

The cumulative distribution function of $\bar{\chi}^2$ is given by:

$$F_{\bar{\chi}^2}(k) = \begin{cases} 1 - \dfrac{\beta^* \, e^{-k/2}}{\pi} - 2\,Q(\sqrt{k}) + 2\displaystyle\int_{\sqrt{k}}^{\infty}\int_{-\infty}^{L_1} Z(x)Z(y)\,dydx, & k \geq 0 \\ 0, & \text{o.w.} \end{cases}$$

where,

$$L_1 = -\frac{1}{a}\left[x + \sqrt{k(1 + a^2)} \right]$$

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Chapter 3

Studying the Conjecture

3.1 Introduction

In this chapter, we study the conjecture that was stated in chapter 1. Specifically, we consider the testing problems

$$P(V_i): H_0: (\theta_1,\theta_2) = (0,0) \quad \text{vs} \quad H_1: (\theta_1,\theta_2) \in V_i - \{(0,0)\}, \tag{3.1.1}$$

where $i = 1,2,3,$

$$V_1 = V_{\pi/2} = \{ (\Delta,\gamma): 0 \le \gamma \le \pi/2 \},$$

$$V_2 = \{ (\Delta,\gamma): \Delta \ge 0, 0 \le \gamma \le 2\pi \},$$

and

$$V_3 = \{ (\Delta,\gamma): \Delta \ge 0, 0 \le \gamma \le \pi/2 \}.$$

Here, notice that $V_3 \subseteq V_1 \subseteq V_2$.

Assume that ϕ_{v_1}, ϕ_{v_2}, ϕ_{v_3} are the LRT's corresponding to $P(V_1)$, $P(V_2)$ and $P(V_3)$, respectively. The aim of this chapter is to see whether there is domination among these tests and to see which one is admissible

Section 3.2 is concerned with the construction of a test that is admissible for the testing problem $P(V_3)$ and dominate ϕ_{V1}. We use the method of Al-Rawwash and Marden (1988) to construct such a test.

Section 3.3 obtains the relation between the LRT's ϕ_{v_1} and ϕ_{v_2}. Also, a power comparison between these two tests for different cone angles, is given in this section.

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The main topic in section 3.4 is to make a power comparison among nine competitive tests. We give the acceptance regions for these tests as well as the relations connecting each of these test with the level of significance.

The following section constructs a test which is admissible for the testing problem $P(V_3)$ and strictly dominates ϕ_{V_1}

3.2 Dominating the LRT in the case of two sided alternative .

In this section, we use the method of Al-Rawwash and Marden (1988) to construct a test which is admissible for the testing problem $P(V_3)$ and dominates the LRT for the testing problem $P(V_1)$, where $P(V_1)$ and $P(V_3)$ are given in (3.1.1). Their method can be summarized as follows

Assume that (X,Y) is random vector defined over $\mathbb{R}^2$, also let $S = X$, $T = Y$, and define a test ϕ^* as

$$\phi^* = \begin{cases} 0, & s < a(t) \\ \gamma(t), & s = a(t), \\ 1 & o.w \end{cases} \qquad (3.2.1)$$

where,

(s,t) is the realization of (S,T)

The main idea of the construction is to take a test ϕ_0 which is not of the form (3.2.1), and find a test ϕ_1 of the form (3.2.1) which dominates ϕ_0 strictly. The test ϕ_1 is the unique test ϕ^* which is of the form (3.2.1) and satisfies:

$$E_0(\phi_0(S,T) \mid T = t) = E_0(\phi^*(S,T) \mid T = t)$$

The basis of the construction is that either X or Y could

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play the role of S as well as any linear combination of X and Y. Now, assume that the set A_1 is the acceptance region for the test ϕ_1, this test can be written in the following form:

$$\phi_1 = \begin{cases} 0, & x \in A_1 \\ 1, & x \notin A_1 \end{cases}$$

The test ϕ_1 may also violate (3.2.1) for a particular choice of s. If this happens, we will obtain a test ϕ_2 of the form (3.2.1) which strictly dominates ϕ_1, the test ϕ_2 may also violate (3.2.1) for another choice of s. Hence, we can get a test ϕ_3 which strictly dominates ϕ_2. Continue the process until we obtain a test which can no longer be improved in this manner.

Now, it can be easily seen from (2.2.2) and figure (2.2.2) that the acceptance region of the LRT $\phi_{V_1} = \phi_0$ can be written in the form

$$A_0 = \left\{ (x,y) : \begin{array}{ll} -d \le x \le \sqrt{d^2 - y^2} & if \quad 0 \le y \le d \\ -\sqrt{d^2 - y^2} \le x \le d & if \quad -d \le y \le 0 \end{array} \right.,$$

Notice that ϕ_{V_1} is not of the form (3.2.1) for any choice of s.

Now, let S = X, T = Y and conditioning on Y, we want to find a function $h_1(y)$ such that

$$P(X \le h_1(y)) = \begin{cases} P(-d \le x \le \sqrt{d^2 - y^2}) & if \quad 0 \le y \le d \\ P(-\sqrt{d^2 - y^2} \le x \le d) & if \quad -d \le y \le 0 \end{cases},$$

which is equivalent to

$$N(h_1(y)) = N(d) + N(\sqrt{d^2 - y^2}) - 1$$

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Therefore,

$$N^{-1}(N(x) + 1 - N(d)) = \sqrt{d^2 - y^2}$$

But, if we assume that

$$W(x) = N^{-1}(N(x) + 1 - N(d)) \qquad (3.2.2)$$

then the previous equation can be written as

$$W^2(x) + y^2 = d^2$$

Therefore, the acceptance region for the constructed test is given by

$$A_1 = \left\{ (x,y): W^2(x) + y^2 \le d^2 \right\}$$

which implies that the LRT for the testing problem $P(V_1)$ after the conditioning can be written as

$$\phi_1(x,y) = \begin{cases} 0, & W^2(x) + y^2 \le d^2 \\ 1, & W^2(x) + y^2 \ge d^2 \end{cases} , \qquad (3.2.3)$$

The acceptance region A_1 is illustrated in figure 3.2.1.

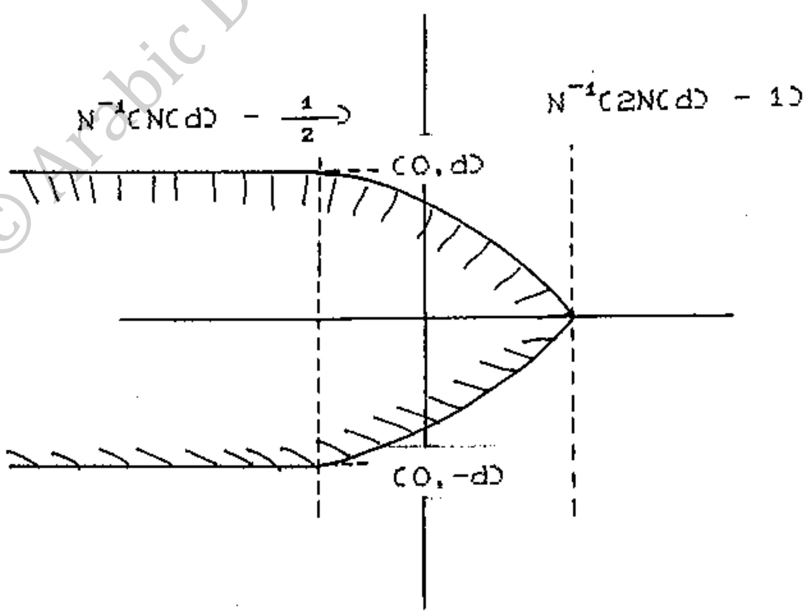

Figure 3.2.1 : Acceptance region A_1

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Notice that $\phi_1(x,y)$ is of the form (3.2.1) for $S = X$ and $T = Y$, but, it is not of the form (3.2.1) for $S = Y$ and $T = X$.

Furthermore, the acceptance region for $\phi_1(x,y)$ can be rewritten in the following form

$$A_1 = \left\{(x,y) : \begin{array}{ll} -d \le y \le d & \textit{if} \quad x \le p \\ -\sqrt{d^2 - W^2(x)} \le y \le \sqrt{d^2 - W^2(x)} & \textit{if} \quad p \le x \le q \end{array}\right. ,$$

where,

$$p = N^{-1}(N(d) - \tfrac{1}{2}) \quad \text{and} \quad q = N^{-1}(2\, N(d) - 1).$$

Applying the construction again by conditioning on X, we find a function $h_2(x)$ such that

$$P(Y \le h_2(y)) = \begin{cases} P(\ -d \le y \le d\ ) & \textit{if} \quad x \le p \\ P(-\sqrt{d^2 - W^2(x)} \le x \le \sqrt{d^2 - W^2(x)} & \textit{if}\ p \le x \le q \end{cases} ,$$

Now, if $x \le p$, we get

$$N(h_2(x)) = 2\, N(d) - 1$$

which implies that

$$h_2(x) = N^{-1}(2\, N(d) - 1) = q.$$

Also, if $p \le x \le q$, we get

$$N(h_2(x)) = 2\, N(\sqrt{d^2 - W^2(x)}\ ) - 1$$

which is equivalent to

$$N^{-1}(\frac{N(y) + 1}{2}) = \sqrt{d^2 - W^2(x)}$$

and this can be simplified as

$$W^2(x) + U^2(y) = d^2$$

where,

$$U(y) = N^{-1}(\frac{N(y) + 1}{2}), \qquad (3.2.4)$$

thus, the acceptance region for the constructed test is given b

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$$A_2 = \left\{ (x,y) : W^2(x) + U^2(y) \le d^2 \right\}$$

Hence, the constracted test can be written as

$$\phi_2(x,y) = \begin{cases} 0, & W^2(x) + U^2(y) \le d^2 \\ 1, & W^2(x) + U^2(y) \ge d^2 \end{cases}, \qquad (3.2.5).$$

The acceptance region for this test is shown in figure 3.2.2.

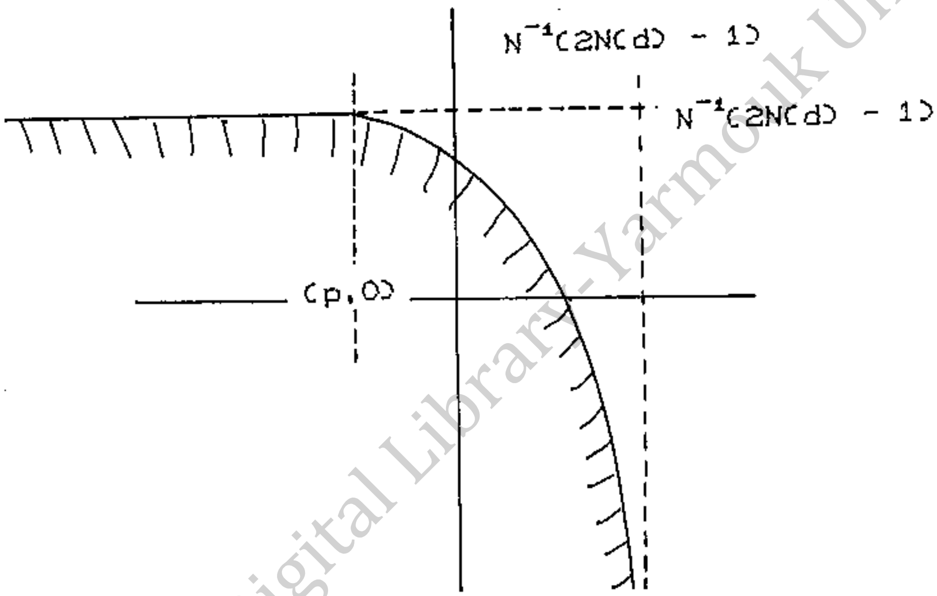

Figure 3.2.2 : Acceptance Region A_2

In order to show that the test $\phi_2(x,y)$ is admissible, we utilize theorem 1.3.1, which, simply, says that if the acceptance region of the test $\phi_2(x,y)$ is decreasing and convex then this test is admissible for testing hypotheses $P(V_3)$.

<u>Theorem 3.2.1</u>

The test $\phi_2(x,y)$ which is given by (3.2.5) is admissible for the testing problem $P(V_1)$ given in (3.1.1), and strictly dominates ϕ_{V_2} .

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Proof:

Applying theorem 1.3.1, it is required to show that the acceptance region for $\phi_2(x,y)$ is convex and decreasing. In order to prove the convexity, we show that $U^2(y)$ and $W^2(x)$ are convex function, also, we show that the function

$$h(x) = N^{-1}(\sqrt{d^2 - W^2(x)})$$

has a maximum point at $x = p$.

Let $C(x) = W^2(x)$, then

$$\frac{\partial C(x)}{\partial x} = 2\, W(x)\, W'(x) \qquad (3.2.6)$$

and

$$\frac{\partial^2 C(x)}{\partial x^2} = 2\Big[\, W(x)\, W''(x) + [\, W(x)]^2\Big]$$

But, from equation (3.2.2), and by taking the first derivative, we get

$$Z(w)\frac{\partial W}{\partial x} = Z(x),$$

therefore

$$\frac{\partial W}{\partial x} = \frac{Z(x)}{Z(w)} \qquad (3.2.7)$$

Also, by taking the second derivative, we get

$$Z(w)\frac{\partial^2 W}{\partial x^2} - w\, Z(w)\left[\frac{\partial W}{\partial x}\right]^2 = -x\, Z(x)$$

therefore,

$$\frac{\partial^2 W(x)}{\partial x^2} = \frac{Z(x)}{Z^2(w)}\Big[w\, Z(x) - x\, Z(w)\Big]$$

$$= e^{\frac{w^2}{2} - x^2}\Big[w\, e^{\frac{w^2}{2}} - x\, e^{\frac{x^2}{2}}\Big]$$

Since $w \geq x$ we have $e^{\frac{w^2}{2}} \geq e^{\frac{x^2}{2}}$, therefore

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$$W''(x) = \frac{\partial^2 W(x)}{\partial x^2} \geq 0$$

which means that $W^2(x)$ is convex. To show that $U^2(y)$ is convex see Jararha (1991). It is enough to show that the function $h(x) = N^{-1}(\sqrt{d^2 - W^2(x)}\,)$ has a maximum at $x = p$.

Now, $N(y) = \sqrt{d^2 - W^2(x)}$

By taking the first derivative we get

$$Z(y)\ \frac{\partial y}{\partial x} = \frac{-2W(x)W'(x)}{\sqrt{d^2 - W^2(x)}}$$

Since $W(p) = 0$, we have

$$\left.\frac{\partial y}{\partial x}\right|_{x = p} = 0$$

Now, by taking the second derivative we get

$$\frac{\partial^2 y}{\partial x^2} = \frac{-2\sqrt{d^2 - W^2(x)}\left[W'^2(x) + W(x)W''(x)\right] - \dfrac{2W(x)W'(x)}{\sqrt{d^2 - W^2(x)}}}{(d^2 - W^2(x))}$$

It can be seen easily that $W'(p) = Z(p)$, therefore

$$\frac{\partial^2 y}{\partial x^2} = -\frac{2\ Z^2(p)}{d} \leq 0$$

This shows that the acceptance region of the test $\phi_2(x,y)$ is convex. It is left to show that this acceptance region is decreasing. In order to accomplish this we have to show that $W^2(x)$ and $U^2(y)$ are increasing. From (3.2.6) and (3.2.7) we have

$$\frac{\partial C(x)}{\partial x} \geq 0$$

which means that $W^2(x)$ is an increasing function of x

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Now,

$$\frac{\partial U^2(y)}{\partial y} = 2U(y)U'(y), \tag{3.2.8}$$

however, from (3.2.4) we have

$$N(U(y)) = \frac{N(y) + 1}{2}$$

and by taking the first derivative, we get

$$Z(U(y)) \frac{\partial U(y)}{\partial y} = \frac{1}{2} Z(y)$$

therefore,

$$\frac{\partial U(y)}{\partial y} = \frac{Z(y)}{2\, Z(U(y))} \geq 0$$

Hence, from (3.2.8) we can see that $U^2(y)$ is an increasing function of y.

To show that the acceptance region A_2 is decreasing, let $(x,y) \in A_2$ and suppose that $x' \leq x$, $y' \leq y$. Its enough to show that $(x',y') \in A_2$. But, $(x,y) \in A_2$ means that $W^2(x) + U^2(y) < d^2$ and because $W^2(x)$, $U^2(y)$ are increasing function, we have

$$W^2(x') \leq W^2(x)$$

and

$$U^2(y') \leq U^2(y)$$

therefore,

$$W^2(x') + U^2(y') \leq W^2(x) + U^2(y)$$

which means that $(x',y') \in A_2$ and this completes the proof.

3.3 Numerical Comparison between LRT's $\phi_{v_{\beta^*}}$ and ϕ_{v_2}

In this section, we give the conditions on the LRT's ϕ_{v_1} and ϕ_{v_2} so that they have the same level of significance. Also,

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we give a comparison between the powers of these LRT's for different cone angles β^*. Furthermore a power comparison is made for different choice of the alternative (θ_1, θ_2).

Lemma 3.3.1

A necessary condition for a test ϕ to dominate another test ϕ^* is that they have the same level of significance, i.e.

$$E_{(0,0)}\phi = E_{(0,0)}\phi^*.$$

Proof:

Let $h(\theta_1, \theta_2) = E_{(\theta_1,\theta_2)}\phi - E_{(\theta_1,\theta_2)}\phi^*$

by using definition 1.3.1, the test ϕ dominates the test ϕ^* if

$$E_{(\theta_1,\theta_2)}\phi \le E_{(\theta_1,\theta_2)}\phi^*, \qquad (\theta_1, \theta_2) = (0,0)$$

and

$$E_{(\theta_1,\theta_2)}\phi \ge E_{(\theta_1,\theta_2)}\phi^*, \qquad \forall\ (\theta_1, \theta_2) \in V_i - \{(0,0)\},$$

with strict inequality for some $(\theta_1, \theta_2) \in V_i - \{(0,0)\}$.

Since, $h(\theta_1, \theta_1)$ is a continuous function in (θ_1, θ_2), therefore

$h(0,0) = 0$. This completes the proof.

It is known from theorem 2.2.2 that the tests ϕ_{v_1} and ϕ_{v_2} has an acceptance region of the form $\{ \bar{\chi}^2 \le d \}$ and $\{ \chi^2 \le c \}$ respectively, where $\bar{\chi}^2$ is given in (2.2.2) and χ^2 is the usual chi-square statistic. The following theorem gives the relation between c and d.

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Theorem 3.3.1:

Consider the testing problems given in (3.1.1). If the tests ϕ_{v_1} and ϕ_{v_2} have the same level of significance, then c and d are related by

$$1 - e^{-c^2/2} = \frac{1}{2}(1 - e^{-d^2/2}) + 2\,(N(d) - \frac{1}{2})^2.$$

Proof:

Assume that A_1 and A_2 are the acceptance regions of ϕ_{v_1} and ϕ_{v_2} which are illustrated in Figure (3.3.1). Moreover,

$$P_{(0,0)}(A_2) = E_{(0,0)}\,\phi_{v_2} = \int_{-c}^{c}\int_{-\sqrt{c^2-y^2}}^{\sqrt{c^2-y^2}} Z(x)\,Z(y)\,dx\,dy$$

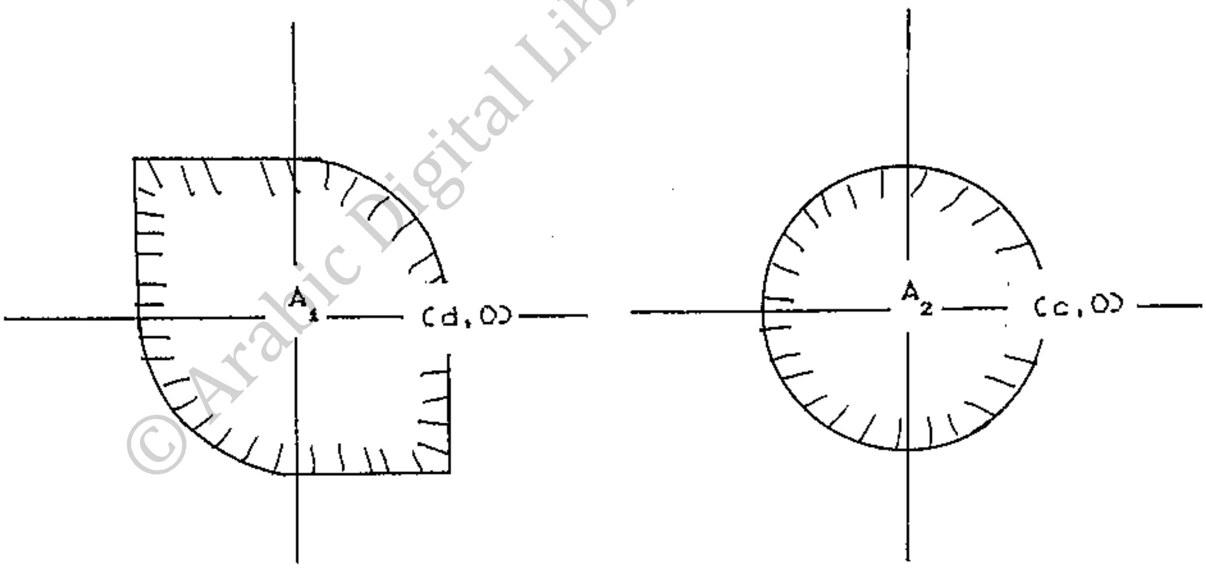

Figure 3.3.1 : Acceptance Region A_1 and A_2

Now, let (r,ξ) be the polar transform of (x,y), then

$$P_{(0,0)}(A_2) = \frac{1}{2\pi}\int_0^{2\pi}\int_0^{c} r\,e^{-r^2/2}\,dr\,d\xi = 1 - e^{-c^2/2} \quad (3.3.1)$$

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In order to find $P_{(0,0)}(A_1)$ we must partition the acceptance region of the LRT ϕ_{V_1} into four subregions $A_1^{(1)}, A_1^{(2)}, A_1^{(3)}$ and $A_1^{(4)}$ which are illustrated in Figure (3.3.2).

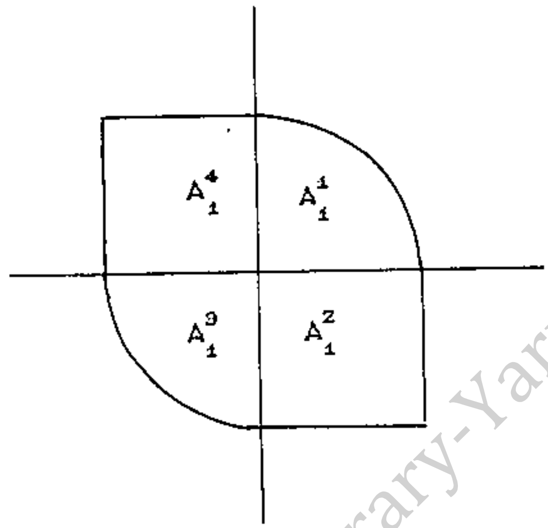

Figure 3.3.2 : Subregions of the acceptance region A_1

Now,

$$P_{(0,0)}(A_1) = \sum_{i=1}^{4} P_{(0,0)}(A_1^{(i)}).$$

$$P(A_1^{(1)}) = \int_0^{\pi/2}\int_0^{d} \frac{r\, e^{-r^2/2}}{2\pi}\, dr\, d\xi = \frac{1}{4}(1 - e^{-d^2/2})$$

Similarly, for the other regions we get

$$P(A_1^{(2)}) = (N(d) - 1/2)^2, \quad P(A_1^{(3)}) = \frac{1}{4}(1 - e^{-d^2/2})$$

and $P(A_1^{(4)}) = (N(d) - 1/2)^2$. Therefore,

$$P_{(0,0)}(A_1) = \frac{1}{2}(1 - e^{-d^2/2}) + 2(N(d) - 1/2)^2 \qquad (3.3.2)$$

By Lemma 3.3.1 and equations (3.3.1) and (3.3.2). Therefore, the conclusion follows.

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The appendix contains two Tables: Table 1 consists of nine column $\beta_1, \beta_2, \ldots, \beta_9$ which gives the power of nine different tests $\phi_1, \phi_2, \ldots, \phi_9$ to be defined later. The power values are obtained for different choices of (Δ, γ) by numerical integration for the density function over the rejection region. In computing these power values the accuracy was up to 10^{-4}.

Table 2 contains the power values corresponding to the LRT $\phi_{V_{\beta^*}}$ for different cone angles β^*. These power values are calculated at the same point used in Table 1, also the accuracy was the same.

In the remaining of this section we will make a comparison between the power values of the LRT's ϕ_{V_1} and ϕ_{V_2} the first two column of Table 1 β_1 and β_2 represents the power values corresponding to the testing problem $P(V_1)$ and $P(V_2)$ respectively. Now, from this table 1 we can see that for all $\Delta \geq 0$, ϕ_{V_1} is more powerful than ϕ_{V_2} for all $\frac{3\pi}{30} \leq \gamma \leq \frac{12\pi}{30}$, but for all $\frac{\pi}{30} \leq \gamma \leq \frac{3\pi}{30}$ and $\frac{12\pi}{30} \leq \gamma \leq \frac{14\pi}{30}$, neither ϕ_{V_1} nor ϕ_{V_2} dominate the other for all $\Delta \geq 0$ (in terms of power values), but, ϕ_{V_2} dominates ϕ_{V_1} for all $\Delta \geq 0$ and $\gamma = 0$ or $\gamma = \frac{\pi}{2}$.

It can be easily seen from the same table that the LRT ϕ_{V_2} dose not depend on γ, also the LRT ϕ_{V_1} is an increasing function of γ for all $\gamma \leq \frac{\pi}{4}$, and a decreasing function for all γ belonging to the interval $(\frac{\pi}{4}, \frac{\pi}{2})$ which support theorem 2.3.3. Similarly Table 1 shows that ϕ_{V_1} is symmetric

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about $\gamma = \frac{\pi}{4}$ which we have proved in theorem 2.3.2.

On the other hand, Table 2 gives the values of the power values for the LRT's $\phi_{V_{\beta^*}}$ and ϕ_{V_2} for different cone angles β^*. Now, table 2 shows that the LRT $\phi_1(x,y)$ dominates $\phi_2(x,y)$ for all (Δ,γ) in the case that the angle of the cone is $\beta^* \le .405\pi$ Also, from the same table it can be seen that the power of the test is a decreasing function of β^*, i.e.
if we have two cones with angles β_1^* and β_2^* and tests ϕ_1 and ϕ_2 respectively where $\beta_1^* \le \beta_2^*$ then the power function for the test ϕ_1 is greater than the power function for the test ϕ_2.

3.4 Power Comparison of Nine Competitive Tests

This section contains a power comparison among nine competitive tests. For each test, we give the acceptance region as well as the relation that connects each test with its level of significance.

The first test is the LRT corresponding to $P(V_1)$ which is a special case from the test given in (2.2.2), and its acceptance region is shown in figure 3.3.1 . The second one is the LRT corresponding to $P(V_2)$ which can be written as

$$\phi_2 = \begin{cases} 1, & x^2 + y^2 \ge c^2 \\ 0, & \text{otherwise} \end{cases},$$

the acceptance region is illustrated in figure 3.3.1 .

The third test is given in (3.2.3), the acceptance region is illustrated in figure 3.2.1. This test is constructed using

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the idea that we make a conditioning on Y for the LRT corresponding to $P(V_1)$. The fourth one also, is given in (3.2.5) but this is constructed by making double conditioning, the first one on Y and the second on X for the LRT corresponding to $P(V_1)$, it's acceptance region is shown in figure 3.2.2.

The fifth and the sixth tests was derived by Al-Rawwash (1986). The relation connecting each test with the level of significance can be written as:

$$Q(b) + \frac{1}{4} e^{-b^2/2} = \alpha,$$

$$Q(g) + \frac{1}{2} e^{-g^2/2} = \alpha,$$

respectively, and the two tests can be written as

$$\phi_5 = \begin{cases} 1, & \bar{\chi}_1^2 \geq b \\ 0, & \bar{\chi}_1^2 \leq b \end{cases}$$

$$\phi_6 = \begin{cases} 1, & \bar{\chi}_2^2 \geq g \\ 0, & \bar{\chi}_2^2 \leq g \end{cases}$$

where,

$$\bar{\chi}_1^2 = \begin{cases} x^2 + y^2 & x \geq 0 \text{ and } y \geq 0 \\ x^2 & x \geq 0 \text{ and } y \leq 0 \\ y^2 & x \leq 0 \text{ and } y \geq 0 \\ 0 & \text{O.W} \end{cases}$$

$$\bar{\chi}_2^2 = \begin{cases} x^2 + y^2 & x \geq 0 \\ y^2 & x \leq 0 \end{cases},$$

$Q(x) = 1 - N(x)$ and α is the level of significance.

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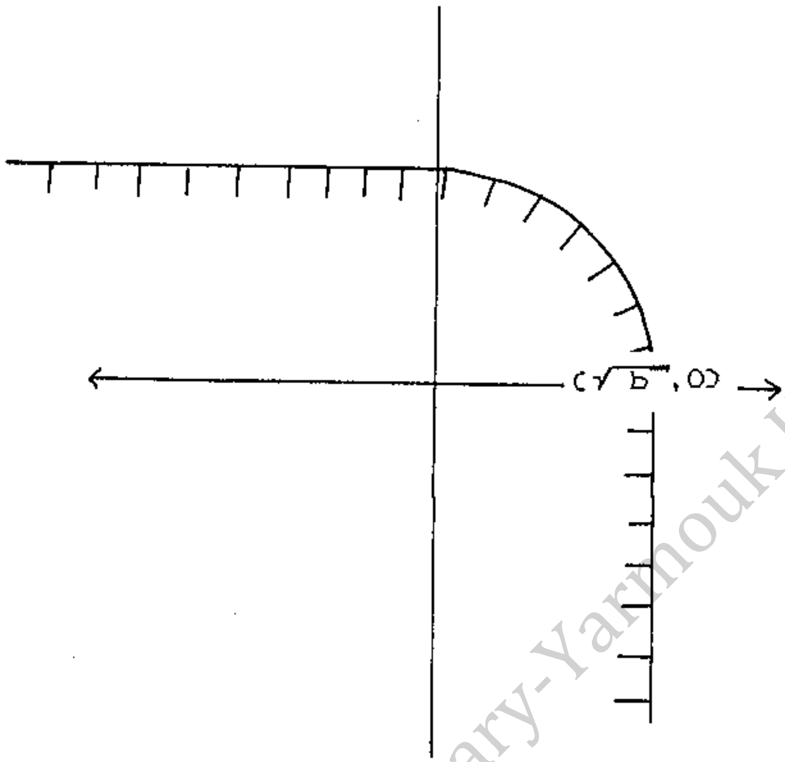

Figure 3.4.1: Acceptance region for the test ϕ_5

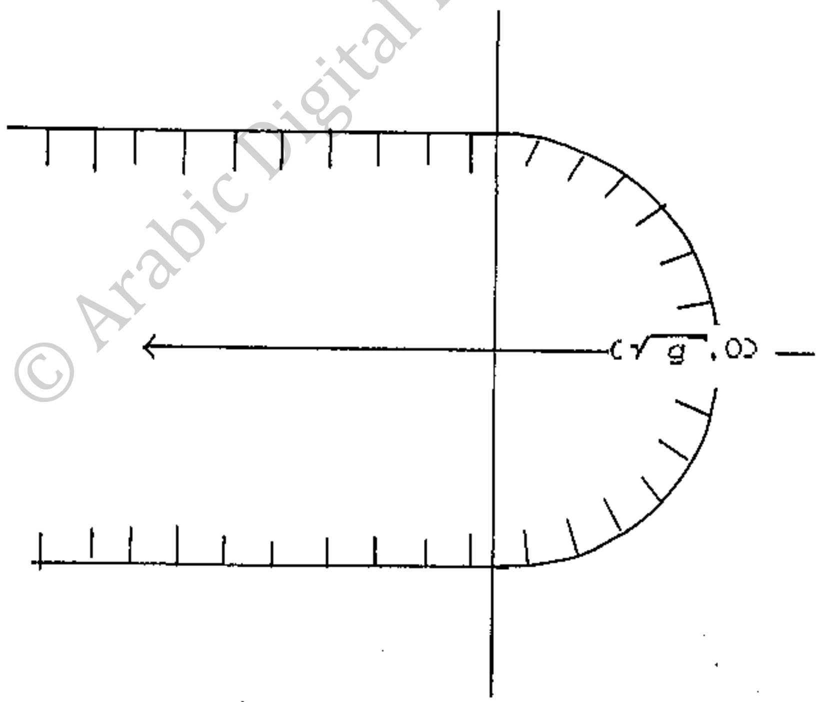

Figure 3.4.2 : Acceptance region for the test ϕ_6

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Tests number seven is constructed by making conditioning on Y for the acceptance region corresponding to the LRT for the testing problem $P(V_2)$. On the contrary, test number eight is constructed by make double conditioning first on Y and the other on X for the LRT corresponding to the testing problem $P(V_2)$, These tests can be written as:

$$\phi_7(x,y) = \begin{cases} 0, & U^2(x) + y^2 \le c^2 \\ 1, & U^2(x) + y^2 \ge c^2 \end{cases},$$

and

$$\phi_8(x,y) = \begin{cases} 0, & U^2(x) + U^2(y) \le c^2 \\ 1, & U^2(x) + U^2(y) \ge c^2 \end{cases},$$

where,

$$U(x) = N^{-1}\left(\frac{N(x) + 1}{2}\right)$$

The acceptance region correspnoding to these two tests is illustrated in figure (3.4.3) and (3.4.4) respectively.

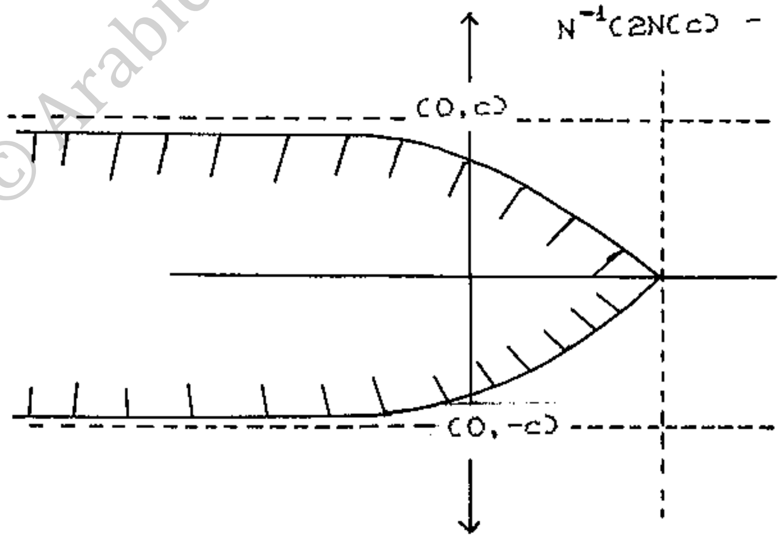

Figure 3.4.3 : Acceptance region for the test ϕ_7

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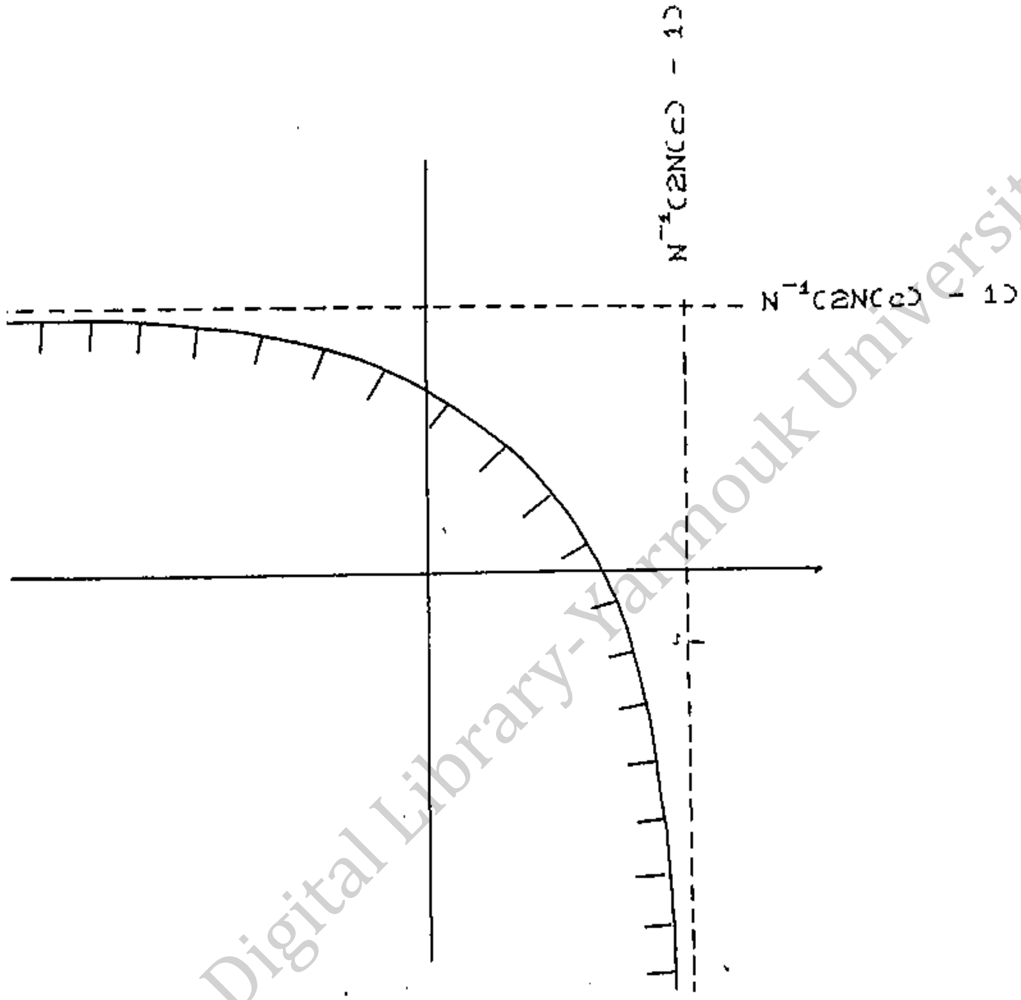

Figure 3.4.4 : acceptance region for the test ϕ_8

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The last test $\phi_9(x,y)$ is constructed by making conditioning on X for the test $\phi_6(x,y)$. The acceptance region of this test is illustrated in figure (3.4.5).

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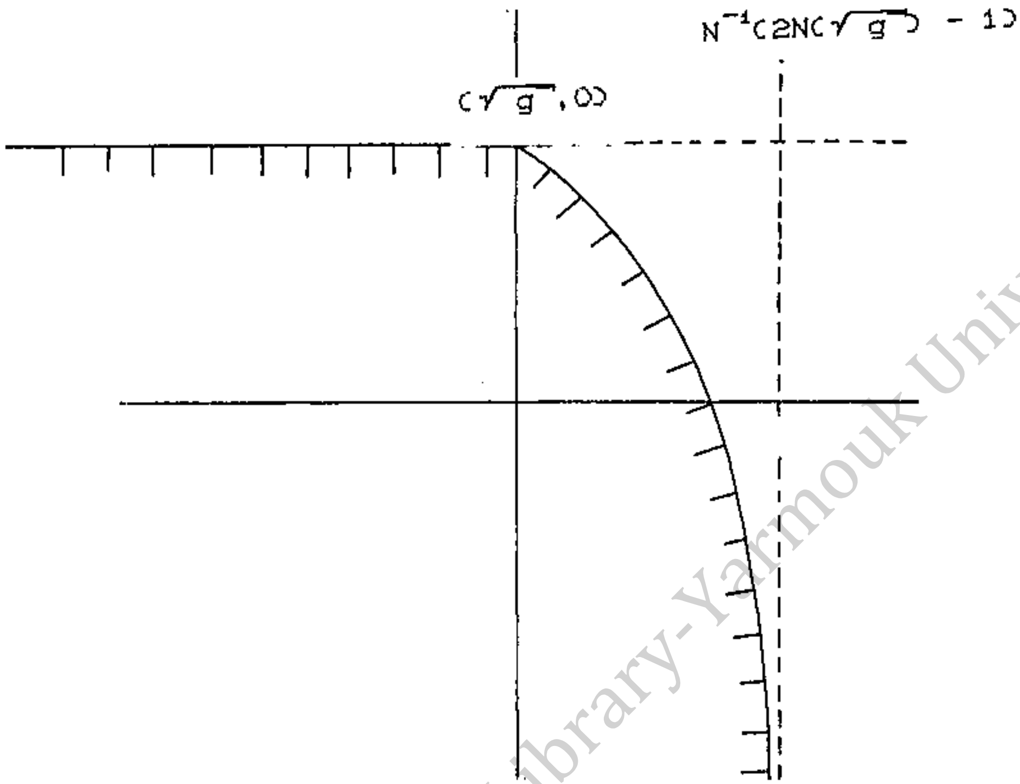

Figure 3.4.5 : Acceptance region for the test ϕ_9

Table 1 contains the power values for these nine tests. From this table, it can be easily seen that for all $\Delta \geq 0$ $\phi_1(x,y)$ is more powerful than $\phi_2(x,y)$ for all $\gamma \in (\frac{\pi}{10}, \frac{4\pi}{10})$ which we discussed it in section 3.3. Also the test $\phi_4(x,y)$ dominates $\phi_3(x,y)$ and both of them dominates $\phi_1(x,y)$ which support the result given in section 3.2, also from the same table it is clear that $\phi_3(x,y)$ is an increasing function of Δ but a decreasing function of γ . On the other hand, $\phi_4(x,y)$ is an increasing function in both Δ and γ.

From the same table, we can see that the test $\phi_9(x,y)$ dominates $\phi_6(x,y)$. This result can be proved in the same manner as in section 3.2. Also the test $\phi_5(x,y)$ dominates $\phi_6(x,y)$ in terms of power values and is dominated by $\phi_9(x,y)$.

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Notice that all these tests are an increasing function of Δ, but, $\phi_6(x,y)$ is a decreasing function of γ and both $\phi_5(x,y)$ and $\phi_9(x,y)$ are an increasing function of γ, $\forall\ \gamma \in (0,\frac{\pi}{4})$ and decreases on the interval $(\frac{\pi}{4},\frac{\pi}{2})$. Moreover, $\phi_5(x,y)$ and $\phi_9(x,y)$ are symmetric about $\gamma = \frac{\pi}{4}$. All this support the results of Al-Rawwash (1986).

Finally, with respect to the tests $\phi_7(x,y)$ and $\phi_8(x,y)$ it can be easily seen that the test $\phi_8(x,y)$ dominates $\phi_7(x,y)$ and these results support the conjecture of Al-Rawwash and Marden (1988). Now, both tests are an increasing function of Δ, on the contrary, $\phi_7(x,y)$ is a decreasing function of γ and $\phi_8(x,y)$ is an increasing function for $\gamma \in (0,\frac{\pi}{4})$ but decreases on the interval $(\frac{\pi}{4},\frac{\pi}{2})$.

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Appendix

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Table 1

Power comparison of Nine Competitive Tests

$\gamma = 0$

Δ	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	β_9
0	.0500	.0500	.0500	.0500	.0500	.0500	.0500	.0501	.0502
.1	.0506	.0507	.0573	.0573	.0568	.0562	.0572	.0573	.0588
.2	.0528	.0530	.0659	.0659	.0651	.0635	.0657	.0657	.0682
.3	.0565	.0568	.0759	.0759	.0747	.0720	.0756	.0756	.0791
.4	.0617	.0622	.0875	.0875	.0860	.0819	.0869	.0869	.0916
.5	.0685	.0693	.1009	.1009	.0990	.0933	.0999	.1000	.1059
.6	.0770	.0781	.1161	.1161	.1138	.1064	.1148	.1148	.1221
.7	.0873	.0888	.1333	.1333	.1307	.1213	.1315	.1315	.1403
.8	.0995	.1014	.1527	.1527	.1497	.1381	.1502	.1502	.1606
.9	.1137	.1160	.1741	.1742	.1709	.1569	.1710	.1710	.1831
1.0	.1300	.1327	.1979	.1979	.1943	.1778	.1939	.1940	.2077
1.1	.1483	.1516	.2238	.2238	.2200	.2009	.2190	.2190	.2345
1.2	.1688	.1728	.2518	.2518	.2478	.2261	.2461	.2461	.2644
1.3	.1914	.1961	.2820	.2820	.2777	.2533	.2753	.2753	.2944
1.4	.2161	.2217	.3140	.3140	.3096	.2826	.3063	.3063	.3271
1.5	.2428	.2495	.3478	.3478	.3432	.3138	.3390	.3390	.3615
1.6	.2713	.2793	.3831	.3831	.3783	.3468	.3732	.3733	.3973
1.7	.3014	.3110	.4196	.4197	.4147	.3812	.4087	.4087	.4342
1.8	.3329	.3444	.4571	.4571	.4519	.4168	.4452	.4452	.4718
1.9	.3654	.3793	.4950	.4950	.4898	.4534	.4823	.4823	.5099
2.0	.3986	.4114	.5332	.5332	.5280	.4906	.5195	.5197	.5480

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Table 1 (cont.)

$\gamma = \pi/30$

Δ	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	β_9
0	.0500	.0500	.0500	.0500	.0500	.0500	.0500	.0501	.0502
.1	.0507	.0507	.0573	.0580	.0575	.0562	.0572	.0580	.0594
.2	.0530	.0530	.0658	.0672	.0664	.0634	.0656	.0672	.0694
.3	.0568	.0568	.0758	.0780	.0768	.0719	.0755	.0779	.0811
.4	.0623	.0622	.0874	.0904	.0888	.0818	.0868	.0903	.0944
.5	.0694	.0693	.1007	.1046	.1027	.0932	.0998	.1044	.1095
.6	.0784	.0781	.1159	.1208	.1185	.1063	.1146	.1203	.1266
.7	.0891	.0888	.1331	.1390	.1362	.1212	.1313	.1383	.1457
.8	.1019	.1014	.1524	.1593	.1562	.1380	.1500	.1583	.1669
.9	.1166	.1160	.1738	.1819	.1784	.1569	.1708	.1804	.1903
1.0	.1335	.1327	.1975	.2066	.2027	.1778	.1937	.2047	.2159
1.1	.1525	.1516	.2233	.2336	.2294	.2008	.2188	.2312	.2436
1.2	.1736	.1728	.2513	.2627	.2581	.2260	.2459	.2597	.2733
1.3	.1969	.1961	.2814	.2938	.2889	.2533	.2750	.2902	.3050
1.4	.2222	.2217	.3134	.3268	.3216	.2826	.3060	.3225	.3385
1.5	.2494	.2495	.3471	.3614	.3559	.3138	.3388	.3564	.3735
1.6	.2783	.2793	.3824	.3974	.3916	.3467	.3730	.3918	.4098
1.7	.3087	.3110	.4188	.4345	.4285	.3812	.4085	.4282	.4471
1.8	.3403	.3444	.4562	.4724	.4662	.4168	.4450	.4654	.4850
1.9	.3726	.3793	.4941	.5107	.5043	.4534	.4821	.5030	.5232
2.0	.4054	.4114	.5322	.5490	.5423	.4906	.5195	.5408	.5613

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Table 1 (cont.)

$\gamma = 2\pi/30$

Δ	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	β_9
0	.0500	.0500	.0500	.0500	.0500	.0500	.0500	.0501	.0502
.1	.0507	.0507	.0572	.0588	.0581	.0561	.0571	.0586	.0599
.2	.0531	.0530	.0656	.0684	.0676	.0633	.0655	.0685	.0705
.3	.0571	.0568	.0755	.0798	.0786	.0717	.0752	.0800	.0827
4	.0628	.0622	.0870	.0929	.0914	.0816	.0865	.0932	.0967
.5	.0703	.0693	.1002	.1079	.1059	.0930	.0994	.1082	.1125
.6	.0796	.0781	.1153	.1248	.1259	.1061	.1141	.1252	.1303
.7	.0908	.0888	.1324	.1438	.1410	.1210	.1308	.1442	.1501
.8	.1040	.1014	.1515	.1650	.1617	.1378	.1495	.1654	.1721
.9	.1193	.1160	.1728	.1883	.1847	.1567	.1702	.1887	.1962
1.0	.1366	.1327	.1963	.2139	.2098	.1776	.1931	.2141	.2225
1.1	.1552	.1516	.2220	.2416	.2372	.2007	.2181	.2417	.2509
1.2	.1778	.1728	.2499	.2714	.2666	.2259	.2452	.2714	.2813
1.3	.2016	.1961	.2798	.3033	.2980	.2532	.2744	.3030	.3136
1.4	.2273	.2217	.3116	.3368	.3312	.2825	.3054	.3363	.3475
1.5	.2548	.2495	.3451	.3720	.3660	.3137	.3381	.3712	.3829
1.6	.2840	.2793	.3802	.4085	.4021	.3467	.3723	.4073	.4195
1.7	.3144	.3110	.4165	.4459	.4392	.3811	.4078	.4443	.4569
1.8	.3459	.3444	.4537	.4840	.4770	.4167	.4443	.4820	.4948
1.9	.3780	.3793	.4915	.5223	.5152	.4533	.4814	.5200	.5330
2.0	.4103	.4114	.5295	.5606	.5533	.4905	.5189	.5579	.5709

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Table 1 (cont.)

$\gamma = 3\pi/30$

Δ	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	β_9
0	.0500	.0500	.0500	.0500	.0500	.0500	.0500	.0501	.0502
.1	.0508	.0507	.0570	.0590	.0586	.0560	.0569	.0591	.0603
.2	.0532	.0530	.0653	.0694	.0686	.0630	.0651	.0697	.0714
.3	.0574	.0568	.0750	.0814	.0802	.0714	.0747	.0818	.0841
.4	.0634	.0622	.0863	.0951	.0935	.0812	.0859	.0958	.0985
.5	.0711	.0693	.0994	.1106	.1087	.0926	.0987	.1116	.1149
.6	.0807	.0781	.1143	.1282	.1258	.1057	.1134	.1294	.1333
.7	.0922	.0888	.1312	.1478	.1450	.1206	.1300	.1493	.1537
.8	.1058	.1014	.1501	.1696	.1663	.1375	.1486	.1714	.1762
.9	.1215	.1160	.1712	.1938	.1899	.1563	.1693	.1956	.2009
1.0	.1393	.1327	.1945	.2197	.2156	.1773	.1921	.2221	.2277
1.1	.1592	.1516	.2199	.2480	.2438	.2004	.2171	.2506	.2566
1.2	.1813	.1728	.2475	.2783	.2734	.2256	.2441	.2812	.2874
1.3	.2054	.1961	.2777	.3105	.3052	.2530	.2732	.3136	.3200
1.4	.2314	.2217	.3086	.3445	.3387	.2823	.3042	.3477	.3543
1.5	.2591	.2495	.3419	.3800	.3738	.3136	.3369	.3832	.3899
1.6	.2883	.2793	.3767	.4166	.4101	.3465	.3712	.4199	.4266
1.7	.3188	.3110	.4127	.4542	.4473	.3809	.4067	.4574	.4641
1.8	.3501	.3444	.4497	.4923	.4851	.4166	.4432	.4954	.5020
1.9	.3819	.3793	.4873	.5306	.5232	.4532	.4803	.5336	.5400
2.0	.4136	.4114	.5252	.5787	.5612	.4904	.5178	.5715	.5777

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Table 1 (cont.)

$\gamma = 4\pi/30$

Δ	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	β_9
0	.0500	.0500	.0500	.0500	.0500	.0500	.0500	.0501	.0502
.1	.0508	.0507	.0567	.0594	.0590	.0558	.0567	.0596	.0606
.2	.0534	.0530	.0648	.0702	.0694	.0627	.0647	.0706	.0720
.3	.0577	.0568	.0743	.0826	.0815	.0710	.0741	.0833	.0851
.4	.0638	.0622	.0854	.0968	.0953	.0807	.0851	.0978	.0999
.5	.0717	.0693	.0982	.1128	.1109	.0921	.0978	.1143	.1167
.6	.0816	.0781	.1129	.1308	.1285	.1051	.1123	.1328	.1355
.7	.0935	.0888	.1295	.1509	.1482	.1200	.1288	.1534	.1563
.8	.1074	.1014	.1482	.1731	.1700	.1369	.1473	.1762	.1792
.9	.1233	.1160	.1690	.1975	.1940	.1588	.1679	.2013	.2043
1.0	.1415	.1327	.1920	.2240	.2201	.1768	.1906	.2284	.2315
1.1	.1617	.1516	.2171	.2527	.2483	.2000	.2155	.2577	.2607
1.2	.1840	.1728	.2443	.2833	.2786	.2252	.2425	.2890	.2918
1.3	.2083	.1961	.2735	.3158	.3106	.2526	.2716	.3220	.3247
1.4	.2345	.2217	.3047	.3500	.3444	.2820	.3025	.3567	.3591
1.5	.2623	.2495	.3376	.3856	.3796	.3133	.3353	.3927	.3948
1.6	.2916	.2793	.3721	.4223	.4160	.3463	.3695	.4298	.4316
1.7	.3220	.3110	.4078	.4598	.4532	.3807	.4050	.4676	.4691
1.8	.3531	.3444	.4445	.4978	.4909	.4164	.4415	.5058	.5069
1.9	.3846	.3793	.4818	.5360	.5289	.4530	.4787	.5440	.5448
2.0	.4156	.4114	.5195	.5740	.5667	.4902	.5163	.5820	.5824

Table 1 (cont.)

$\gamma = 5\pi/30$

Δ	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	β_9
0	.0500	.0500	.0500	.0500	.0500	.0500	.0500	.0501	.0502
.1	.0508	.0507	.0564	.0597	.0593	.0558	.0564	.0599	.0608
.2	.0534	.0530	.0642	.0708	.0700	.0623	.0641	.0713	.0724
.3	.0579	.0568	.0734	.0835	.0824	.0704	.0733	.0844	.0857
.4	.0641	.0622	.0842	.0980	.0966	.0800	.0841	.0994	.1008
.5	.0722	.0693	.0968	.1134	.1126	.0913	.0965	.1164	.1178
.6	.0823	.0781	.1112	.1327	.1306	.1044	.1109	.1354	.1369
.7	.0944	.0888	.1275	.1531	.1506	.1193	.1272	.1566	.1580
.8	.1085	.1014	.1458	.1756	.1728	.1362	.1488	.1799	.1812
.9	.1247	.1160	.1663	.2002	.1970	.1551	.1660	.2055	.2065
1.0	.1431	.1327	.1888	.2270	.2234	.1761	.1886	.2332	.2339
1.1	.1635	.1516	.2135	.2559	.2519	.1994	.2134	.2630	.2633
1.2	.1860	.1728	.2403	.2867	.2823	.2247	.2404	.2948	.2946
1.3	.2105	.1961	.2692	.3193	.3145	.2521	.2694	.3283	.3276
1.4	.2367	.2217	.2999	.3535	.3484	.2815	.3003	.3634	.3622
1.5	.2646	.2495	.3324	.3891	.3837	.3129	.3330	.3997	.3980
1.6	.2939	.2793	.3665	.4258	.4200	.3459	.3672	.4371	.4347
1.7	.3242	.3110	.4019	.4632	.4572	.3804	.4027	.4751	.4722
1.8	.3551	.3444	.4382	.5011	.4949	.4161	.4393	.5134	.5100
1.9	.3864	.3793	.4753	.5391	.5327	.4527	.4765	.5516	.5478
2.0	.4174	.4114	.5128	.5769	.5703	.4900	.5141	.5895	.5853

Table 1 (cont.)

$\gamma = 6\pi/30$

Δ	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	β_9
0	.0500	.0500	.0500	.0500	.0500	.0500	.0500	.0501	.0502
.1	.0508	.0507	.0561	.0598	.0595	.0552	.0560	.0601	.0609
.2	.0535	.0530	.0635	.0711	.0705	.0618	.0634	.0718	.0726
.3	.0580	.0568	.0724	.0840	.0831	.0697	.0723	.0852	.0860
.4	.0643	.0622	.0829	.0987	.0975	.0792	.0828	.1005	.1012
.5	.0726	.0693	.0951	.1153	.1137	.0904	.0950	.1178	.1184
.6	.0828	.0781	.1091	.1338	.1319	.1034	.1091	.1372	.1378
.7	.0950	.0888	.1250	.1544	.1522	.1183	.1252	.1587	.1587
.8	.1093	.1014	.1430	.1771	.1746	.1352	.1434	.1824	.1820
.9	.1257	.1160	.1630	.2019	.1990	.1541	.1637	.2084	.2075
1.0	.1442	.1327	.1852	.2288	.2256	.1752	.1862	.2364	.2350
1.1	.1647	.1516	.2094	.2577	.2542	.1985	.2108	.2666	.2645
1.2	.1873	.1728	.2358	.2885	.2847	.2239	.2376	.2987	.2959
1.3	.2119	.1961	.2642	.3211	.3171	.2514	.2665	.3325	.3290
1.4	.2382	.2217	.2945	.3583	.3510	.2809	.2974	.3678	.3636
1.5	.2651	.2495	.3366	.3908	.3862	.3123	.3300	.4043	.3994
1.6	.2953	.2793	.3602	.4274	.4226	.3453	.3642	.4419	.4362
1.7	.3255	.3110	.3952	.4648	.4597	.3799	.3997	.4800	.4737
1.8	.3564	.3444	.4313	.5025	.4973	.4156	.4363	.5183	.5115
1.9	.3875	.3793	.4681	.5404	.5351	.4523	.4736	.5566	.5493
2.0	.4183	.4114	.5054	.5780	.5726	.4896	.5113	.5944	.5867

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Table 1 (cont.)

$\gamma = 7\pi/30$

Δ	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	β_9
0	.0500	.0500	.0500	.0500	.0500	.0500	.0500	.0501	.0502
.1	.0508	.0507	.0556	.0599	.0596	.0549	.0556	.0602	.0608
.2	.0535	.0530	.0627	.0713	.0707	.0611	.0625	.0720	.0725
.3	.0581	.0568	.0712	.0843	.0834	.0689	.0710	.0856	.0859
.4	.0645	.0622	.0812	.0990	.0979	.0782	.0812	.1010	.1011
.5	.0728	.0693	.0930	.1156	.1143	.0893	.0935	.1185	.1182
.6	.0831	.0781	.1067	.1342	.1326	.1022	.1068	.1380	.1373
.7	.0953	.0888	.1222	.1548	.1530	.1170	.1226	.1598	.1585
.8	.1097	.1014	.1397	.1775	.1755	.1339	.1400	.1837	.1818
.9	.1261	.1160	.1593	.2023	.2000	.1528	.1603	.2098	.2072
1.0	.1447	.1327	.1810	.2292	.2267	.1740	.1829	.2380	.2347
1.1	.1653	.1516	.2048	.2581	.2554	.1972	.2078	.2683	.2642
1.2	.1880	.1728	.2307	.2889	.2859	.2227	.2334	.3006	.2956
1.3	.2126	.1961	.2586	.3215	.3183	.2503	.2628	.3345	.3287
1.4	.2389	.2217	.2885	.3556	.3522	.2798	.2929	.3700	.3633
1.5	.2668	.2495	.3202	.3910	.3875	.3113	.3250	.4006	.3992
1.6	.2960	.2793	.3535	.4275	.4238	.3444	.3599	.4442	.4360
1.7	.3262	.3110	.3881	.4647	.4609	.3791	.3956	.4824	.4736
1.8	.3570	.3444	.4239	.5024	.4985	.4149	.4318	.5208	.5114
1.9	.3880	.3793	.4605	.5401	.5362	.4516	.4695	.5592	.5492
2.0	.4188	.4114	.4976	.5776	.5736	.4890	.5069	.5970	.5867

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Table 1 (cont)

$\gamma = 8\pi/30$

Δ	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	β_9
0	.0500	.0500	.0500	.0500	.0500	.0500	.0500	.0501	.0502
.1	.0508	.0507	.0552	.0599	.0596	.0545	.0552	.0602	.0607
.2	.0535	.0530	.0617	.0712	.0707	.0604	.0617	.0721	.0723
.3	.0581	.0568	.0698	.0841	.0834	.0679	.0698	.0856	.0855
.4	.0645	.0622	.0794	.0988	.0979	.0771	.0796	.1011	.1005
.5	.0728	.0693	.0908	.1154	.1143	.0880	.0912	.1185	.1174
.6	.0831	.0781	.1039	.1339	.1326	.1007	.1046	.1380	.1364
.7	.0953	.0888	.1190	.1544	.1530	.1155	.1201	.1598	.1574
.8	.1097	.1014	.1361	.1770	.1754	.1323	.1377	.1837	.1805
.9	.1261	.1160	.1552	.2017	.2000	.1512	.1574	.2098	.2068
1.0	.1447	.1327	.1764	.2285	.2267	.1723	.1794	.2380	.2331
1.1	.1653	.1516	.1997	.2573	.2554	.1956	.2036	.2683	.2626
1.2	.1880	.1728	.2252	.2880	.2860	.2211	.2300	.3006	.2938
1.3	.2126	.1961	.2527	.3205	.3183	.2487	.2588	.3345	.3269
1.4	.2389	.2217	.2821	.3545	.3522	.2784	.2891	.3700	.3614
1.5	.2668	.2495	.3134	.3898	.3878	.3099	.3215	.4007	.3973
1.6	.2960	.2793	.3463	.4262	.4239	.3431	.3556	.4443	.4342
1.7	.3262	.3110	.3807	.4633	.4610	.3778	.3910	.4824	.4718
1.8	.3570	.3444	.4162	.5009	.4985	.4138	.4276	.5208	.5097
1.9	.3880	.3793	.4526	.5386	.5362	.4506	.4680	.5592	.5476
2.0	.4188	.4114	.4897	.5760	.5736	.4880	.5028	.5970	.5852

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Table 1 (cont)

$\gamma = 9\pi/30$

Δ	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	β_9
0	.0500	.0500	.0500	.0500	.0500	.0500	.0500	.0501	.0502
.1	.0508	.0507	.0546	.0597	.0595	.0541	.0546	.0601	.0604
.2	.0535	.0530	.0606	.0709	.0705	.0596	.0607	.0718	.0718
.3	.0580	.0568	.0682	.0837	.0831	.0667	.0683	.0852	.0846
.4	.0643	.0622	.0774	.0982	.0975	.0757	.0777	.1005	.0993
.5	.0726	.0693	.0883	.1145	.1137	.0863	.0888	.1178	.1160
.6	.0828	.0781	.1009	.1328	.1319	.0988	.1018	.1372	.1347
.7	.0950	.0888	.1155	.1531	.1522	.1129	.1169	.1587	.1554
.8	.1093	.1014	.1320	.1756	.1746	.1300	.1341	.1824	.1783
.9	.1257	.1160	.1507	.2001	.1990	.1488	.1534	.2083	.2025
1.0	.1442	.1327	.1714	.2267	.2256	.1699	.1750	.2364	.2300
1.1	.1647	.1516	.1942	.2553	.2542	.1931	.1989	.2665	.2590
1.2	.1873	.1728	.2192	.2858	.2847	.2186	.2250	.2986	.2900
1.3	.2119	.1961	.2462	.3181	.3171	.2462	.2532	.3324	.3228
1.4	.2382	.2217	.2753	.3520	.3510	.2759	.2835	.3678	.3570
1.5	.2661	.2495	.3062	.3872	.3862	.3075	.3157	.4044	.3925
1.6	.2953	.2793	.3388	.4235	.4226	.3407	.3496	.4419	.4290
1.7	.3255	.3110	.3729	.4606	.4597	.3755	.3850	.4801	.4650
1.8	.3564	.3444	.4083	.4982	.4973	.4116	.4215	.5185	.5015
1.9	.3875	.3793	.4446	.5359	.5351	.4485	.4589	.5568	.5403
2.0	.4183	.4114	.4816	.5733	.5726	.4861	.4967	.5946	.5897

Table 1 (cont)

$\gamma = 10\pi/30$

A	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	β_9
0	.0500	.0500	.0500	.0500	.0500	.0500	.0500	.0501	.0502
.1	.0508	.0507	.0541	.0595	.0593	.0536	.0541	.0599	.0601
.2	.0534	.0530	.0595	.0704	.0700	.0588	.0596	.0714	.0710
.3	.0579	.0568	.0666	.0829	.0824	.0656	.0667	.0845	.0838
.4	.0641	.0622	.0752	.0971	.0966	.0742	.0755	.0995	.0978
.5	.0722	.0693	.0855	.1131	.1126	.0846	.0862	.1164	.1139
.6	.0823	.0781	.0976	.1311	.1306	.0969	.0987	.1354	.1321
.7	.0944	.0888	.1117	.1511	.1506	.1113	.1133	.1566	.1523
.8	.1085	.1014	.1277	.1731	.1727	.1278	.1300	.1799	.1746
.9	.1247	.1160	.1457	.1974	.1970	.1465	.1488	.2055	.1991
1.0	.1431	.1327	.1659	.2237	.2234	.1674	.1700	.2332	.2258
1.1	.1635	.1516	.1883	.2520	.2519	.1906	.1934	.2630	.2545
1.2	.1860	.1728	.2128	.2832	.2823	.2161	.2190	.2947	.2853
1.3	.2108	.1961	.2394	.3144	.3145	.2437	.2468	.3283	.3179
1.4	.2367	.2217	.2681	.3581	.3484	.2734	.2768	.3634	.3521
1.5	.2646	.2495	.2986	.3832	.3837	.3050	.3087	.3997	.3878
1.6	.2939	.2793	.3310	.4195	.4200	.3384	.3423	.4372	.4246
1.7	.3242	.3110	.3649	.4566	.4572	.3733	.3775	.4752	.4622
1.8	.3551	.3444	.4001	.4942	.4949	.4094	.4139	.5136	.5002
1.9	.3864	.3793	.4364	.5319	.5327	.4465	.4512	.5519	.5384
2.0	.4174	.4114	.4734	.5695	.5704	.4842	.5891	.5898	.5763

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Table 1 (cont.)

$\gamma = 11\pi/30$

Δ	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	β_9
0	.0500	.0500	.0500	.0500	.0500	.0500	.0500	.0501	.0502
.1	.0508	.0507	.0534	.0591	.0590	.0531	.0535	.0596	.0597
.2	.0534	.0530	.0583	.0697	.0694	.0578	.0581	.0707	.0701
.3	.0577	.0568	.0648	.0817	.0815	.0642	.0655	.0834	.0821
.4	.0638	.0622	.0728	.0955	.0953	.0724	.0731	.0979	.0957
.5	.0717	.0693	.0825	.1111	.1109	.0825	.0832	.1144	.1113
.6	.0816	.0781	.0941	.1286	.1285	.0945	.0948	.1328	.1288
.7	.0935	.0888	.1075	.1482	.1482	.1086	.1103	.1535	.1483
.8	.1074	.1014	.1229	.1698	.1700	.1248	.1258	.1762	.1700
.9	.1233	.1160	.1404	.1936	.1940	.1433	.1433	.2012	.1939
1.0	.1415	.1327	.1600	.2195	.2201	.1640	.1641	.2284	.2200
1.1	.1617	.1516	.1818	.2475	.2483	.1870	.1870	.2576	.2482
1.2	.1840	.1728	.2058	.2774	.2785	.2123	.2118	.2889	.2784
1.3	.2083	.1961	.2320	.3093	.3106	.2398	.2395	.3220	.3108
1.4	.2345	.2217	.2603	.3428	.3444	.2694	.2692	.3567	.3444
1.5	.2623	.2495	.2905	.3778	.3796	.3010	.2995	.3927	.3797
1.6	.2916	.2793	.3226	.4139	.4160	.3344	.3323	.4289	.4163
1.7	.3220	.3110	.3563	.4510	.4532	.3693	.3675	.4677	.4538
1.8	.3531	.3444	.3914	.4887	.4909	.4055	.4034	.5060	.4918
1.9	.3846	.3793	.4277	.5265	.5289	.4427	.4410	.5443	.5300
2.0	.4156	.4114	.4647	.5643	.5667	.4805	.5785	.5822	.5680

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Table 1 (cont.)

$\gamma = 12\pi/30$

Δ	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	β_9
0	.0500	.0500	.0500	.0500	.0500	.0500	.0500	.0501	.0502
.1	.0508	.0507	.0528	.0587	.0586	.0526	.0528	.0592	.0591
.2	.0532	.0530	.0571	.0688	.0686	.0568	.0572	.0697	.0689
.3	.0574	.0568	.0629	.0803	.0802	.0628	.0631	.0819	.0803
.4	.0634	.0622	.0703	.0935	.0935	.0708	.0707	.0958	.0933
.5	.0711	.0693	.0793	.1086	.1087	.0801	.0801	.1116	.1081
.6	.0807	.0781	.0902	.1255	.1258	.0917	.0914	.1294	.1248
.7	.0922	.0888	.1030	.1445	.1450	.1054	.1047	.1493	.1435
.8	.1058	.1014	.1178	.1656	.1663	.1213	.1210	.1414	.1644
.9	.1215	.1160	.1346	.1888	.1899	.1394	.1376	.1956	.1875
1.0	.1393	.1327	.1537	.2141	.2156	.1598	.1575	.2220	.2127
1.1	.1592	.1516	.1749	.2416	.2435	.1825	.1795	.2505	.2401
1.2	.1813	.1728	.1983	.2712	.2734	.2075	.2039	.2811	.2696
1.3	.2054	.1961	.2239	.3027	.3052	.2347	.2305	.3135	.3011
1.4	.2314	.2217	.2517	.3359	.3387	.2641	.2593	.3476	.3344
1.5	.2591	.2495	.2816	.3706	.3738	.2956	.2902	.3832	.3692
1.6	.2883	.2793	.3133	.4067	.4101	.3289	.3229	.4200	.4054
1.7	.3188	.3110	.3468	.4437	.4473	.3637	.3574	.4575	.4425
1.8	.3501	.3444	.3818	.4814	.4851	.4000	.3932	.4956	.4804
1.9	.3819	.3793	.4180	.5194	.5232	.4373	.4301	.5339	.5185
2.0	.4136	.4114	.4552	.5573	.5612	.4752	.5679	.5719	.5566

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Table 1 (cont.)

$\gamma = 13\pi/30$

Δ	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	β_9
0	.0500	.0500	.0500	.0500	.0500	.0500	.0500	.0501	.0502
.1	.0507	.0507	.0521	.0582	.0581	.0520	.0522	.0586	.0585
.2	.0531	.0530	.0557	.0677	.0676	.0557	.0558	.0686	.0676
.3	.0571	.0568	.0609	.0786	.0787	.0612	.0611	.0801	.0782
.4	.0628	.0622	.0676	.0912	.0914	.0684	.0680	.0933	.0904
.5	.0703	.0693	.0760	.1055	.1059	.0775	.0767	.1083	.1043
.6	.0796	.0781	.0862	.1218	.1239	.0886	.0873	.1253	.1201
.7	.0908	.0888	.1098	.1400	.1410	.1018	.0998	.1443	.1379
.8	.1040	.1014	.1123	.1604	.1618	.1172	.1144	.1654	.1578
.9	.1193	.1160	.1284	.1829	.1847	.1348	.1311	.1886	.1798
1.0	.1366	.1327	.1467	.2076	.2098	.1547	.1500	.2141	.2041
1.1	.1562	.1516	.1672	.2344	.2372	.1769	.1712	.2417	.2306
1.2	.1778	.1728	.1900	.2634	.2666	.2014	.1947	.2713	.2590
1.3	.2016	.1961	.2150	.2944	.2980	.2283	.2204	.3030	.2898
1.4	.2273	.2217	.2422	.3272	.3312	.2574	.2483	.3362	.3220
1.5	.2548	.2495	.2716	.3616	.3660	.2885	.2784	.3711	.3560
1.6	.2840	.2793	.3030	.3974	.4021	.3215	.3104	.4073	.3915
1.7	.3144	.3110	.3615	.4343	.4392	.3562	.3442	.4445	.4282
1.8	.3459	.3444	.3709	.4719	.4770	.3923	.3795	.4823	.4656
1.9	.3780	.3793	.4070	.5100	.5152	.4296	.4161	.5203	.5035
2.0	.4103	.4114	.4441	.5408	.5533	.4675	.4535	.5584	.5416

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Table 1 (cont.)

$\gamma = 14\pi/30$

Δ	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	β_9
0	.0500	.0500	.0500	.0500	.0500	.0500	.0500	.0501	.0502
.1	.0507	.0507	.0514	.0576	.0575	.0515	.0515	.0580	.0578
.2	.0530	.0530	.0544	.0664	.0664	.0546	.0545	.0673	.0661
.3	.0568	.0568	.0588	.0766	.0768	.0595	.0590	.0781	.0759
.4	.0623	.0622	.0684	.0884	.0889	.0661	.0652	.0904	.0871
.5	.0694	.0693	.0725	.1020	.1027	.0747	.0731	.1045	.1001
.6	.0784	.0781	.0819	.1174	.1184	.0852	.0892	.1204	.1148
.7	.0891	.0888	.0932	.1348	.1363	.0977	.0945	.1383	.1315
.8	.1019	.1014	.1064	.1543	.1562	.1125	.1081	.1583	.1502
.9	.1166	.1160	.1218	.1789	.1784	.1294	.1238	.1804	.1710
1.0	.1335	.1327	.1392	.1998	.2028	.1486	.1417	.2047	.1940
1.1	.1525	.1516	.1589	.2258	.2294	.1702	.1619	.2311	.2192
1.2	.1736	.1728	.1809	.2540	.2581	.1941	.1842	.2596	.2465
1.3	.1969	.1961	.2051	.2843	.2889	.2203	.2089	.2901	.2758
1.4	.2222	.2217	.2316	.3165	.3215	.2488	.2357	.3224	.3071
1.5	.2494	.2495	.2603	.3504	.3559	.2793	.2648	.3564	.3402
1.6	.2783	.2793	.2910	.3857	.3916	.3119	.2958	.3918	.3747
1.7	.3087	.3110	.3237	.4223	.4285	.3462	.3287	.4283	.4105
1.8	.3403	.3444	.3581	.4598	.4661	.3820	.3632	.4656	.4473
1.9	.3726	.3793	.3938	.4978	.5042	.4190	.3991	.5033	.4848
2.0	.4054	.4114	.4308	.5359	.5424	.4569	.4361	.5412	.5225

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Table 1 (cont.)

$\gamma = \pi/2$

Δ	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	β_9
0	.0500	.0500	.0500	.0500	.0500	.0500	.0500	.0501	.0502
.1	.0506	.0507	.0507	.0569	.0569	.0509	.0508	.0573	.0570
.2	.0528	.0530	.0529	.0649	.0651	.0534	.0531	.0658	.0645
.3	.0565	.0568	.0567	.0744	.0747	.0577	.0569	.0757	.0733
.4	.0617	.0622	.0619	.0854	.0860	.0637	.0623	.0871	.0836
.5	.0685	.0693	.0688	.0980	.0990	.0716	.0694	.1001	.0954
.6	.0770	.0781	.0774	.1125	.1138	.0814	.0782	.1149	.1090
.7	.0873	.0888	.0879	.1289	.1307	.0932	.0889	.1316	.1244
.8	.0995	.1014	.1002	.1473	.1497	.1071	.1014	.1503	.1418
.9	.1137	.1160	.1146	.1679	.1709	.1233	.1160	.1711	.1612
1.0	.1300	.1327	.1311	.1907	.1943	.1416	.1327	.1939	.1827
1.1	.1483	.1516	.1499	.2158	.2200	.1623	.1516	.2189	.2064
1.2	.1688	.1728	.1708	.2430	.2478	.1853	.1727	.2460	.2322
1.3	.1914	.1961	.1941	.2723	.2777	.2106	.1960	.2752	.2601
1.4	.2161	.2217	.2196	.3036	.3096	.2382	.2215	.3062	.2899
1.5	.2428	.2495	.2474	.3367	.3432	.2679	.2492	.3389	.3216
1.6	.2713	.2793	.2773	.3714	.3783	.2997	.2790	.3732	.3548
1.7	.3014	.3110	.3091	.4074	.4146	.3333	.3107	.4088	.3896
1.8	.3329	.3444	.3428	.4445	.4519	.3685	.3441	.4453	.4254
1.9	.3654	.3793	.3781	.4822	.4898	.4051	.3791	.4826	.4621
2.0	.3986	.4114	.4146	.5203	.5280	.4426	.4152	.5201	.4993

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Table 2

Power function for the LRT $\phi_{V_{\beta^*}}$ for different cone angles

$\gamma = 0$

$\Delta \backslash \beta^*$	$\pi/12$	$\pi/6$	$\pi/4$	$\pi/3$	$.405\pi$	$\pi/2$	2π
0	.0500	.0500	.0500	.0500	.0500	.0499	.0500
.1	.0518	.0516	.0514	.0510	.0508	.0506	.0507
.2	.0546	.0543	.0542	.0538	.0531	.0528	.0530
.3	.0607	.0597	.0594	.0585	.0571	.0565	.0568
.4	.0685	.0672	.0653	.0645	.0628	.0617	.0622
.5	.0784	.0767	.0749	.0721	.0701	.0685	.0693
.6	.0912	.0888	.0852	.0822	.0794	.0770	.0781
.7	.1065	.1020	.0981	.0934	.0905	.0873	.0888
.8	.1235	.1199	.1138	.1077	.1036	.0995	.1014
.9	.1436	.1385	.1311	.1231	.1187	.1137	.1160
1.0	.1666	.1593	.1501	.1429	.1361	.1300	.1327
1.1	.1910	.1821	.1723	.1626	.1555	.1483	.1516
1.2	.2182	.2081	.1976	.1851	.1772	.1688	.1728
1.3	.2483	.2366	.2235	.2100	.2010	.1914	.1961
1.4	.2800	.2672	.2516	.2377	.2269	.2161	.2217
1.5	.3131	.2989	.2826	.2655	.2548	.2428	.2495
1.6	.3487	.3329	.3133	.2963	.2844	.2713	.2793
1.7	.3855	.3664	.3476	.3287	.3155	.3014	.3110
1.8	.4223	.4028	.3819	.3611	.3479	.3329	.3444
1.9	.4609	.4381	.4166	.3957	.3812	.3654	.3793
2.0	.4979	.4747	.4511	.4292	.4154	.3986	.4114

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Table 2 (cont.)

$\gamma = \pi/30$

$\Delta \backslash \beta^*$	$\pi/12$	$\pi/6$	$\pi/4$	$\pi/3$	$.405\pi$	$\pi/2$	2π
0	.0500	.0500	.0500	.0500	.0500	.0499	.0500
.1	.0519	.0517	.0515	.0511	.0508	.0507	.0507
.2	.0550	.0546	.0544	.0540	.0532	.0530	.0530
.3	.0609	.0607	.0597	.0586	.0575	.0568	.0568
.4	.0688	.0673	.0662	.0657	.0633	.0623	.0622
.5	.0787	.0779	.0754	.0731	.0706	.0694	.0693
.6	.0915	.0902	.0874	.0844	.0812	.0784	.0781
.7	.1072	.1041	.1001	.0966	.0950	.0891	.0888
.8	.1245	.1214	.1162	.1118	.1100	.1019	.1014
.9	.1453	.1414	.1354	.1284	.1260	.1166	.1160
1.0	.1689	.1633	.1555	.1473	.1449	.1335	.1327
1.1	.1937	.1879	.1788	.1686	.1643	.1525	.1516
1.2	.2211	.2147	.2039	.1921	.1891	.1736	.1728
1.3	.2518	.2422	.2300	.2170	.2138	.1969	.1961
1.4	.2833	.2732	.2595	.2464	.2418	.2222	.2217
1.5	.3170	.3066	.2906	.2741	.2702	.2494	.2495
1.6	.3522	.3391	.3232	.3052	.2999	.2783	.2793
1.7	.3899	.3743	.3569	.3386	.3282	.3087	.3110
1.8	.4260	.4100	.3907	.3781	.3567	.3403	.3444
1.9	.4646	.4466	.4253	.4082	.3989	.3726	.3793
2.0	.5022	.4821	.4608	.4331	.4215	.4054	.4114

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Table 2 (cont.)

$\gamma = 2\pi/30$

$\Delta \backslash \beta^*$	$\pi/12$	$\pi/6$	$\pi/4$	$\pi/3$	$.405\pi$	$\pi/2$	2π
0	.0500	.0500	.0500	.0500	.0500	.0499	.0500
.1	.0519	.0518	.0517	.0511	.0508	.0507	.0507
.2	.0550	.0547	.0545	.0541	.0535	.0531	.0530
.3	.0609	.0609	.0599	.0590	.0582	.0571	.0568
.4	.0688	.0682	.0676	.0659	.0634	.0628	.0622
.5	.0787	.0783	.0762	.0743	.0710	.0703	.0693
.6	.0915	.0909	.0887	.0854	.0813	.0796	.0781
.7	.1072	.1055	.1023	.0982	.0933	.0908	.0888
.8	.1245	.1232	.1198	.1133	.1111	.1040	.1014
.9	.1453	.1427	.1374	.1311	.1243	.1193	.1160
1.0	.1689	.1651	.1597	.1518	.1430	.1366	.1327
1.1	.1937	.1899	.1823	.1737	.1665	.1562	.1516
1.2	.2211	.2164	.2088	.1976	.1859	.1778	.1728
1.3	.2518	.2451	.2359	.2233	.2159	.2016	.1961
1.4	.2833	.2762	.2657	.2511	.2419	.2273	.2217
1.5	.3170	.3099	.2967	.2813	.2702	.2548	.2495
1.6	.3522	.3431	.3296	.3129	.3009	.2840	.2793
1.7	.3899	.3781	.3639	.3459	.3311	.3144	.3110
1.8	.4260	.4147	.3797	.3782	.3597	.3459	.3444
1.9	.4646	.4580	.4311	.4116	.3992	.3780	.3793
2.0	.5022	.4856	.4652	.4451	.4268	.4103	.4114

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Table 2 (cont.)

$\gamma = 3\pi/30$

$\Delta \backslash \beta^*$	$\pi/12$	$\pi/6$	$\pi/4$	$\pi/3$	$.405\pi$	$\pi/2$	2π
0	.0500	.0500	.0500	.0500	.0500	.0499	.0500
.1	.0518	.0518	.0518	.0512	.0509	.0508	.0507
.2	.0547	.0547	.0547	.0541	.0535	.0532	.0530
.3	.0609	.0609	.0608	.0592	.0585	.0574	.0568
.4	.0689	.0682	.0678	.0662	.0642	.0634	.0622
.5	.0789	.0783	.0778	.0759	.0715	.0711	.0693
.6	.0920	.0909	.0897	.0865	.0820	.0807	.0781
.7	.1071	.1055	.1035	.1001	.0941	.0922	.0888
.8	.1242	.1232	.1209	.1154	.1082	.1058	.1014
.9	.1450	.1427	.1390	.1332	.1219	.1215	.1160
1.0	.1673	.1651	.1601	.1547	.1408	.1393	.1327
1.1	.1927	.1899	.1844	.1762	.1677	.1592	.1516
1.2	.2199	.2164	.2105	.2018	.1949	.1813	.1728
1.3	.2498	.2451	.2384	.2276	.2166	.2054	.1961
1.4	.2820	.2762	.2681	.2567	.2467	.2314	.2217
1.5	.3155	.3099	.2999	.2861	.2715	.2591	.2495
1.6	.3510	.3431	.3323	.3176	.3030	.2883	.2793
1.7	.3885	.3781	.3664	.3504	.3393	.3188	.3110
1.8	.4253	.4147	.4001	.3836	.3678	.3501	.3444
1.9	.4640	.4580	.4343	.4164	.3999	.3819	.3793
2.0	.5001	.4856	.4684	.4495	.4285	.4136	.4114

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Table 2 (cont.)

$\gamma = 4\pi/30$

Δ	$\pi/12$	$\pi/6$	$\pi/4$	$\pi/3$	$.405\pi$	$\pi/2$	2π
0		.0500	.0500	.0500	.0500	.0499	.0500
.1		.0517	.0516	.0513	.0509	.0508	.0507
.2		.0546	.0546	.0542	.0536	.0534	.0530
.3		.0607	.0605	.0594	.0586	.0577	.0568
.4		.0673	.0672	.0665	.0650	.0638	.0622
.5		.0779	.0776	.0761	.0725	.0717	.0693
.6		.0902	.0895	.0875	.0833	.0816	.0781
.7		.1041	.1033	.1015	.0944	.0935	.0888
.8		.1214	.1207	.1171	.1112	.1074	.1014
.9		.1414	.1399	.1354	.1227	.1233	.1160
1.0		.1633	.1612	.1558	.1418	.1415	.1327
1.1		.1879	.1856	.1785	.1688	.1617	.1516
1.2		.2147	.2112	.2033	.1967	.1840	.1728
1.3		.2422	.2397	.2300	.2221	.2083	.1961
1.4		.2732	.2698	.2587	.2485	.2345	.2217
1.5		.3066	.3000	.2898	.2729	.2623	.2495
1.6		.3391	.3331	.3204	.3087	.2916	.2793
1.7		.3743	.3676	.3521	.3451	.3220	.3110
1.8		.4100	.4010	.3854	.3686	.3531	.3444
1.9		.4466	.4353	.4184	.4045	.3846	.3793
2.0		.4821	.4682	.4511	.4326	.4156	.4114

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Table 2 (cont)

$\gamma = 5\pi/30$

Δ	π/12	π/6	π/4	π/3	.405π	π/2	2π
0		.0500	.0500	.0500	.0500	.0499	.0500
.1		.0516	.0515	.0514	.0510	.0508	.0507
.2		.0543	.0543	.0542	.0538	.0534	.0530
.3		.0597	.0596	.0595	.0587	.0579	.0568
.4		.0672	.0670	.0669	.0655	.0641	.0622
.5		.0767	.0774	.0764	.0731	.0722	.0693
.6		.0888	.0892	.0878	.0839	.0823	.0781
.7		.1020	.1032	.1018	.0976	.0944	.0888
.8		.1199	.1197	.1175	.1121	.1085	.1014
.9		.1385	.1385	.1358	.1310	.1247	.1160
1.0		.1593	.1607	.1566	.1534	.1431	.1327
1.1		.1821	.1839	.1793	.1696	.1635	.1516
1.2		.2081	.2090	.2045	.1970	.1860	.1728
1.3		.2366	.2377	.2317	.2231	.2105	.1961
1.4		.2672	.2675	.2591	.2489	.2367	.2217
1.5		.2989	.2988	.2899	.2757	.2646	.2495
1.6		.3329	.3311	.3213	.3099	.2939	.2793
1.7		.3664	.3656	.3536	.3465	.3242	.3110
1.8		.4028	.3999	.3862	.3791	.3551	.3444
1.9		.4381	.4332	.4193	.4055	.3864	.3793
2.0		.4747	.4671	.4519	.4331	.4174	.4114

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Table 2 (cont)

$\gamma = 6\pi/30$

Δ	$\pi/12$	$\pi/6$	$\pi/4$	$\pi/3$	$.405\pi$	$\pi/2$	2π
0			.0500	.0500	.0500	.0499	.0500
.1			.0514	.0513	.0509	.0508	.0507
.2			.0543	.0542	.0538	.0535	.0530
.3			.0595	.0594	.0586	.0580	.0568
.4			.0668	.0665	.0653	.0643	.0622
.5			.0769	.0761	.0730	.0726	.0693
.6			.0887	.0873	.0835	.0828	.0781
.7			.1015	.1015	.0959	.0950	.0888
.8			.1182	.1171	.1113	.1093	.1014
.9			.1366	.1354	.1287	.1257	.1160
1.0			.1573	.1558	.1528	.1442	.1327
1.1			.1800	.1785	.1651	.1647	.1516
1.2			.2065	.2033	.1922	.1873	.1728
1.3			.2333	.2300	.2222	.2119	.1961
1.4			.2636	.2587	.2478	.2382	.2217
1.5			.2945	.2898	.2746	.2661	.2495
1.6			.3261	.3204	.3087	.2953	.2793
1.7			.3601	.3521	.3450	.3255	.3110
1.8			.3944	.3854	.3779	.3564	.3444
1.9			.4299	.4184	.4049	.3875	.3793
2.0			.4635	.4511	.4338	.4183	.4114

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Table 2 (cont)

$\gamma = 7\pi/30$

Δ	$\pi/12$	$\pi/6$	$\pi/4$	$\pi/3$	$.408\pi$	$\pi/2$	2π
0			.0500	.0500	.0500	.0499	.0500
.1			.0514	.0512	.0509	.0508	.0507
.2			.0542	.0541	.0535	.0535	.0530
.3			.0593	.0592	.0585	.0581	.0568
.4			.0665	.0662	.0652	.0645	.0622
.5			.0756	.0759	.0729	.0728	.0693
.6			.0867	.0865	.0831	.0831	.0781
.7			.0993	.1001	.0953	.0953	.0888
.8			.1151	.1154	.1105	.1097	.1014
.9			.1333	.1332	.1278	.1261	.1160
1.0			.1539	.1547	.1519	.1447	.1327
1.1			.1755	.1762	.1643	.1653	.1516
1.2			.2001	.2018	.1911	.1880	.1728
1.3			.2272	.2276	.2210	.2126	.1961
1.4			.2564	.2567	.2462	.2389	.2217
1.5			.2867	.2861	.2731	.2668	.2495
1.6			.3181	.3176	.3070	.2960	.2793
1.7			.3524	.3604	.3433	.3262	.3110
1.8			.3863	.3836	.3758	.3570	.3444
1.9			.4212	.4164	.4027	.3880	.3793
2.0			.4565	.4495	.4321	.4188	.4114

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Table 2 (cont)

$\gamma = 8\pi/30$

Δ	$\pi/12$	$\pi/6$	$\pi/4$	$\pi/3$	$.408\pi$	$\pi/2$	2π
0			.0500	.0500	.0500	.0499	.0500
.1			.0514	.0511	.0509	.0508	.0507
.2			.0542	.0541	.0534	.0535	.0530
.3			.0591	.0590	.0583	.0581	.0568
.4			.0656	.0659	.0649	.0645	.0622
.5			.0743	.0743	.0723	.0728	.0693
.6			.0851	.0854	.0821	.0831	.0781
.7			.0989	.0982	.0940	.0953	.0888
.8			.1132	.1133	.1097	.1097	.1014
.9			.1313	.1311	.1269	.1261	.1160
1.0			.1505	.1518	.1501	.1447	.1327
1.1			.1724	.1737	.1637	.1653	.1516
1.2			.1976	.1976	.1902	.1880	.1728
1.3			.2234	.2233	.2199	.2126	.1961
1.4			.2514	.2511	.2451	.2389	.2217
1.5			.2829	.2813	.2722	.2668	.2495
1.6			.3136	.3129	.3065	.2960	.2793
1.7			.3475	.3459	.3427	.3262	.3110
1.8			.3812	.3782	.3743	.3570	.3444
1.9			.4166	.4116	.4011	.3880	.3793
2.0			.4514	.4451	.4308	.4188	.4114

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Table 2 (cont)

$\gamma = 9\pi/30$

Δ	$\pi/12$	$\pi/6$	$\pi/4$	$\pi/3$	$.408\pi$	$\pi/2$	2π
0				.0500	.0500	.0499	.0500
.1				.0511	.0508	.0508	.0507
.2				.0540	.0533	.0535	.0530
.3				.0586	.0580	.0580	.0568
.4				.0657	.0642	.0643	.0622
.5				.0731	.0716	.0726	.0693
.6				.0844	.0814	.0828	.0781
.7				.0966	.0932	.0950	.0888
.8				.1118	.1089	.1093	.1014
.9				.1284	.1257	.1257	.1160
1.0				.1473	.1489	.1442	.1327
1.1				.1686	.1622	.1647	.1516
1.2				.1921	.1888	.1873	.1728
1.3				.2170	.2185	.2119	.1961
1.4				.2454	.2439	.2382	.2217
1.5				.2741	.2709	.2661	.2495
1.6				.3052	.3040	.2953	.2793
1.7				.3386	.3391	.3255	.3110
1.8				.3781	.3712	.3564	.3444
1.9				.4082	.3997	.3875	.3793
2.0				.4331	.4289	.4183	.4114

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Table 2 (cont)

$\gamma = 10\pi/30$

Δ	$\pi/12$	$\pi/6$	$\pi/4$	$\pi/3$	$.408\pi$	$\pi/2$	2π
0				.0500	.0500	.0499	.0500
.1				.0510	.0508	.0508	.0507
.2				.0538	.0532	.0534	.0530
.3				.0585	.0578	.0579	.0568
.4				.0645	.0640	.0641	.0622
.5				.0721	.0713	.0722	.0693
.6				.0822	.0811	.0823	.0781
.7				.0934	.0928	.0944	.0888
.8				.1077	.1085	.1085	.1014
.9				.1231	.1252	.1247	.1160
1.0				.1429	.1481	.1431	.1327
1.1				.1626	.1613	.1635	.1516
1.2				.1851	.1878	.1860	.1728
1.3				.2100	.2164	.2105	.1961
1.4				.2377	.2423	.2367	.2217
1.5				.2655	.2692	.2646	.2495
1.6				.2963	.3030	.2939	.2793
1.7				.3287	.3378	.3242	.3110
1.8				.3611	.3697	.3551	.3444
1.9				.3957	.3980	.3864	.3793
2.0				.4292	.4275	.4174	.4114

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Table 2 (cont.)

$\gamma = 11\pi/30$

Δ	π/12	π/6	π/4	π/3	.405π	π/2	2π
0					.0500	.0499	.0500
.1					.0508	.0508	.0507
.2					.0532	.0534	.0530
.3					.0575	.0577	.0568
.4					.0635	.0638	.0622
.5					.0708	.0717	.0693
.6					.0802	.0816	.0781
.7					.0914	.0935	.0888
.8					.1061	.1074	.1014
.9					.1212	.1233	.1160
1.0					.1413	.1415	.1327
1.1					.1591	.1617	.1516
1.2					.1837	.1840	.1728
1.3					.2091	.2083	.1961
1.4					.2349	.2345	.2217
1.5					.2626	.2623	.2495
1.6					.2936	.2916	.2793
1.7					.3274	.3220	.3110
1.8					.3588	.3531	.3444
1.9					.3897	.3846	.3793
2.0					.4205	.4156	.4114

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Table 2 (cont.)

$\gamma = 12\pi/30$

Δ	$\pi/12$	$\pi/5$	$\pi/4$	$\pi/3$	$.405\pi$	$\pi/2$	2π
0					.0500	.0499	.0500
.1					.0508	.0508	.0507
.2					.0531	.0532	.0530
.3					.0571	.0574	.0568
.4					.0628	.0634	.0622
.5					.0701	.0711	.0693
.6					.0794	.0807	.0781
.7					.0905	.0922	.0888
.8					.1036	.1058	.1014
.9					.1187	.1215	.1160
1.0					.1361	.1393	.1327
1.1					.1555	.1592	.1516
1.2					.1772	.1813	.1728
1.3					.2010	.2054	.1961
1.4					.2269	.2314	.2217
1.5					.2548	.2591	.2495
1.6					.2844	.2883	.2793
1.7					.3155	.3188	.3110
1.8					.3479	.3501	.3444
1.9					.3812	.3819	.3793
2.0					.4154	.4136	.4114

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